Lec 26: Deterministic reduction from PH to #SAT

Recap

Theorem 1: (Randomized reduction from Σ_c-SAT to $\oplus$SAT)

There is a randomized reduction g that given a parameter p and a QBF φ with c levels of alternations, runs in time $\mathrm{poly}(|\varphi|, p)$ and outputs a circuit $g(\varphi)$ s.t.

$$\varphi \text{ is true} \Rightarrow \Pr[\oplus g(\varphi) \text{ is true}] \geq 1 - \frac{1}{2^p},$$

$$\varphi \text{ is false} \Rightarrow \Pr[\oplus g(\varphi) \text{ is false}] \geq 1 - \frac{1}{2^p}.$$

Note: $|g(\varphi)|$ is exponential in c. This is fine as $c = O(1)$.

- **Obs:** A randomized reduction can be viewed as a deterministic reduction that additionally takes a random string as input.
- Set $p = 2$ in Theorem 1.
- **Corollary *:** There is a deterministic reduction g that given a QBF φ with c levels of alternations and a random string $\underline{r} \in \{0,1\}^R$, where $R = |\varphi|^{O(1)}$, runs in time $\mathrm{poly}(|\varphi|)$ and outputs a ckt. $g(\varphi, \underline{r})$ s.t.

$$\varphi \text{ is true} \implies \Pr_{\underline{r} \in_r \{0,1\}^R}\left[\oplus g(\varphi, \underline{r}) \text{ is true}\right] \geq \frac{3}{4},$$

$$\varphi \text{ is false} \implies \Pr_{\underline{r} \in_r \{0,1\}^R}\left[\oplus g(\varphi, \underline{r}) \text{ is false}\right] \geq \frac{3}{4}.$$

Viewing the reduction g as a DTM

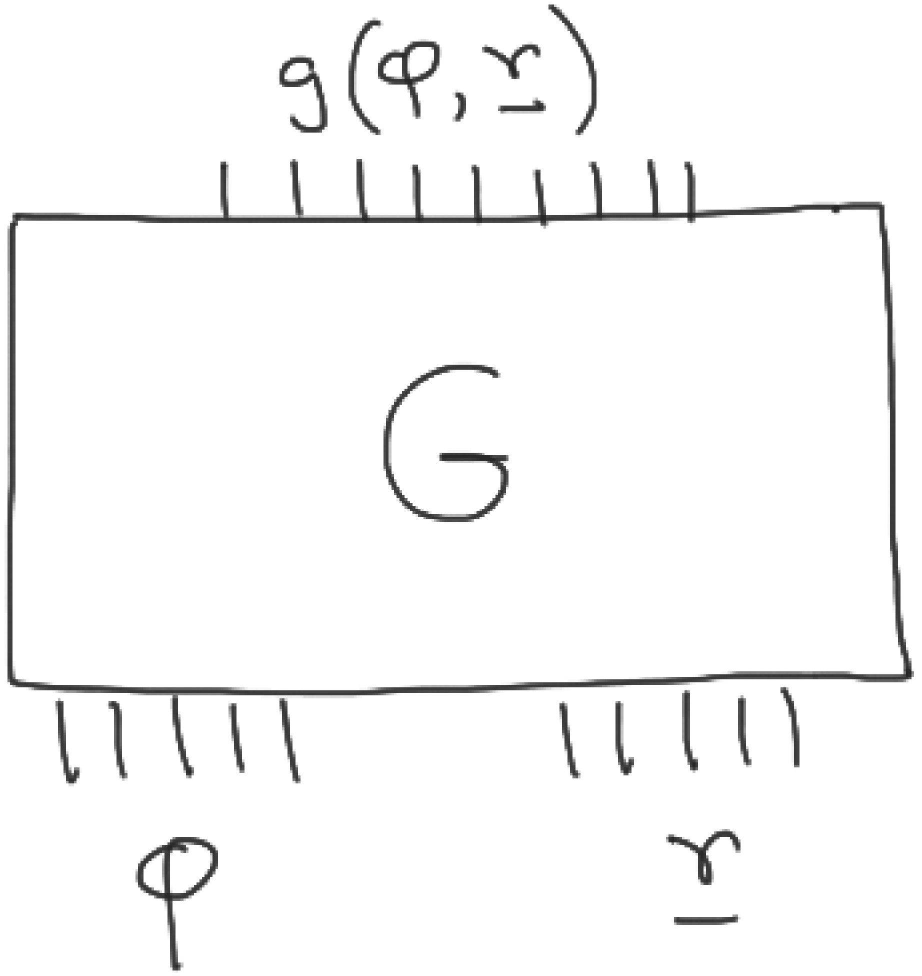


Fig 1: DTM G corresponding to the reduction g.

- Recall Step 2 of the proof of Toda's theorem.

Step 2: (Derandomization of Step 1). Give a deterministic poly-time reduction from PH to #SAT.

- Remark: It would follow from the proof of Step 2 that only one query to the #SAT oracle is sufficient.

- Notations: (a) Let $\varphi_1(\underline{x}_1), \cdots, \varphi_m(\underline{x}_m)$ be ckts. on disjoint sets of variables. Define,

$$(\varphi_1 \cdot \varphi_2 \cdot \cdots \cdot \varphi_m)(\tilde{\underline{x}}) := \varphi_1(\underline{x}_1) \wedge \cdots \wedge \varphi_m(\underline{x}_m).$$

— $|\varphi_1 \cdot \varphi_2 \cdot \cdots \cdot \varphi_m| = |\varphi_1| + \cdots + |\varphi_m| + m$.

— $\#(\varphi_1 \cdot \varphi_2 \cdot \cdots \cdot \varphi_m) = \#\varphi_1 \cdot \#\varphi_2 \cdot \cdots \cdot \#\varphi_m$.

(b) Let $\varphi_1(x_1,\dots,x_{n_1})$ and $\varphi_2(x_1,\dots,x_{n_2})$ be ckts. and $n_2 \geq n_1$.

Define,

$$(\varphi_1+\varphi_2)(z,x_1,\dots,x_{n_2}) := (z \wedge \varphi_2) \vee (\neg z \wedge x_{n_1+1} \wedge \dots \wedge x_{n_2} \wedge \varphi_1).$$

- $|\varphi_1+\varphi_2| = |\varphi_1| + |\varphi_2| + O(n_2)$.
- $\#(\varphi_1+\varphi_2) = \#\varphi_1 + \#\varphi_2$.

(c) For $c \in \mathbb{Z}_{>0}$,

$$\boxed{\begin{aligned} c\varphi &:= \underbrace{\varphi + (\varphi + (\varphi + \dots + (\varphi+\varphi)))}_{c\text{-times}}. \\ \varphi^c &:= \underbrace{\varphi \cdot \varphi \cdot \dots \cdot \varphi}_{c\text{-times}}. \end{aligned}} \quad \text{— (0)}$$

Proof of Step 2

- **Lemma *:** There is a deterministic poly-time reduction that given a parameter $\ell \in \mathbb{Z}_{>0}$ and a ckt. ψ, runs in time $\mathrm{poly}(|\psi|, \ell)$ and outputs a ckt. τ s.t.
  - ψ is true $\implies \#\tau = -1 \mod 2^{\ell+1}$
  - ψ is false $\implies \#\tau = 0 \mod 2^{\ell+1}$.
- **Proof:** The idea is to construct τ iteratively. At the $(i+1)$-th iteration, we construct ψ_{i+1} from ψ_i s.t.

$$\boxed{\begin{aligned} \#\psi_i = -1 \mod 2^{2^i} &\implies \#\psi_{i+1} = -1 \mod 2^{2^{i+1}} \\ \#\psi_i = 0 \mod 2^{2^i} &\implies \#\psi_{i+1} = 0 \mod 2^{2^{i+1}}. \end{aligned}} \quad \text{— (1)}$$

- Finally, set $\Psi_0 = \Psi$ and $\tau = \Psi_{\lceil \log(\ell+1) \rceil}$.
- Let us focus on the construction of Ψ_{i+1} from Ψ_i. Let $\#\Psi_i = t$. We will construct a polynomial $p(t)$ with positive integer coefficients such that

$$\boxed{t = -1 \mod 2^{2^i} \implies t^2 \cdot p(t) = -1 \mod 2^{2^{i+1}}.} \quad \text{—— (2)}$$

- Note: If $t = 0 \mod 2^{2^i}$, then $t^2 = 0 \mod 2^{2^{i+1}}$ and so, $t^2 p(t) = 0 \mod 2^{2^{i+1}}$. Hence, if we define

$$\Psi_{i+1} := \underbrace{\Psi_i^2 \cdot p(\Psi_i)}_{\text{The intepretation of this ckt. is given by Eqn(0).}},$$ then Eqn(1) is satisfied.

- **Choice of $p(t)$:** Set $\boxed{p(t) = 3t^2 + 4t}$.
- **Obs:** $t = -1 \bmod 2^{2^i} \implies t^2 \cdot p(t) = -1 \bmod 2^{2^{i+1}}$.

Proof: Let $t = k \cdot 2^{2^i} - 1$ for $k \in \mathbb{Z}$.

$$\Rightarrow t^2 = -(2k\,2^{2^i} - 1) \bmod 2^{2^{i+1}} .$$

$$\Rightarrow 3t^2 + 4t = -6k\,2^{2^i} + 3 + 4k \cdot 2^{2^i} - 4 \quad \bmod 2^{2^{i+1}}$$

$$= -(2k\,2^{2^i} + 1) \quad \bmod 2^{2^{i+1}} .$$

$$\Rightarrow t^2 \cdot p(t) = (2k\,2^{2^i} - 1)(2k\,2^{2^i} + 1) \quad \bmod 2^{2^{i+1}} ,$$

$$= -1 \quad \bmod 2^{2^{i+1}} .$$

$\blacksquare$

- Therefore, $\psi_{i+1} = \psi_i^2 \cdot p(\psi_i) = \psi_i^2 \cdot (3\psi_i^2 + 4\psi_i)$
$$= 3\psi_i^4 + 4\psi_i^3 .$$
- Note: $|\psi_{i+1}| = |3\psi_i^4 + 4\psi_i^3| = \Theta(|\psi_i^4|) = \Theta(|\psi_i|)$.

$$\therefore \; |\tau| = |\psi_{\lceil \log(\ell+1) \rceil}| = \mathrm{poly}(\ell, |\psi|). \qquad \blacksquare \text{ Lemma } *$$

- Let us denote the deterministic reduction in Lemma $*$ by h.
- Now think of composing h with the reduction g in Corollary $*$ by setting $\ell = R$ (where R is as in Cor $*$).
- Let $\gamma := h(g(\varphi, \underline{r}))$. Denote the vars. of γ by $\underline{\omega}$.

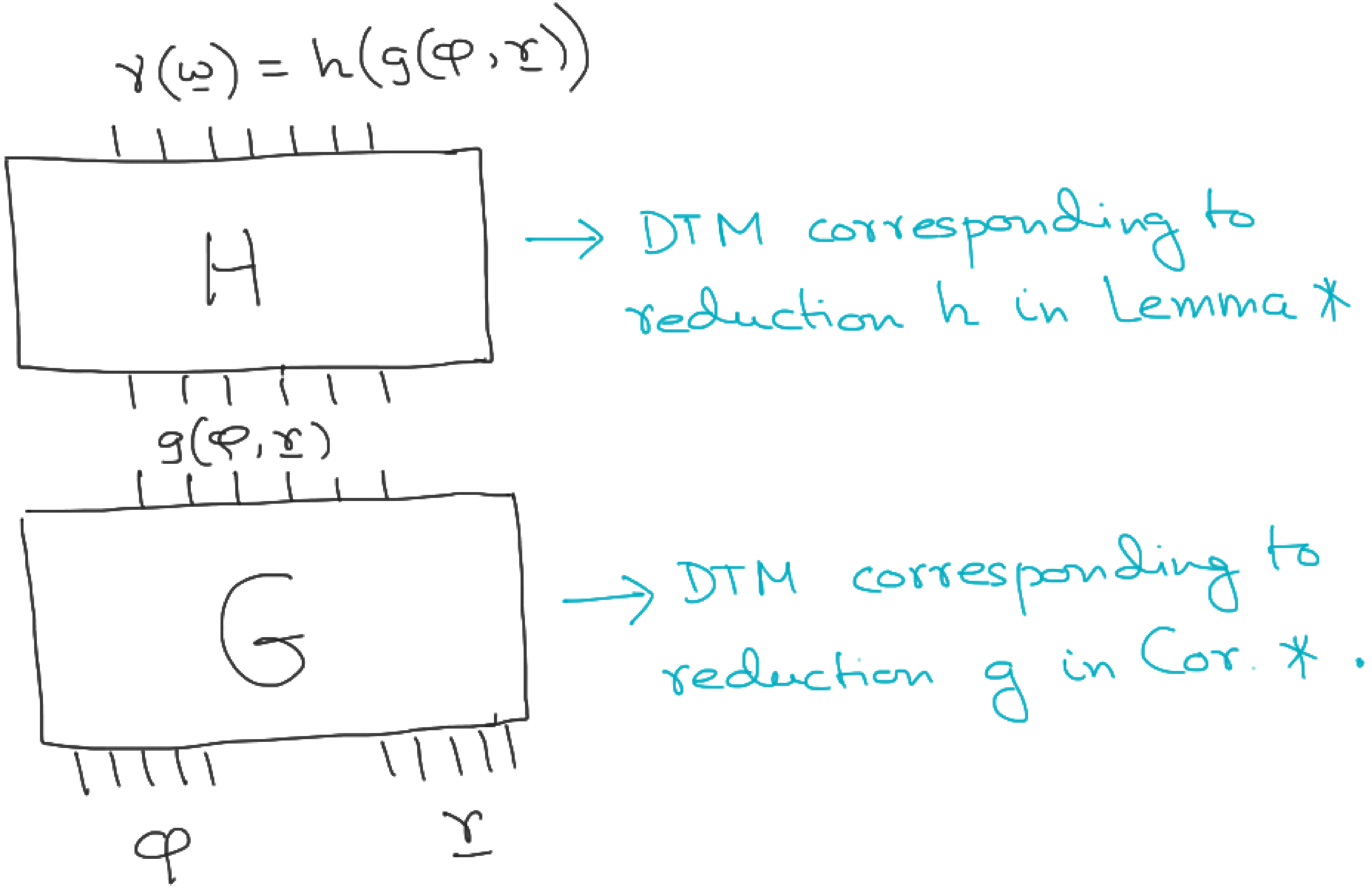


Fig 2: Composing the two reductions g and h

- Consider the following sum

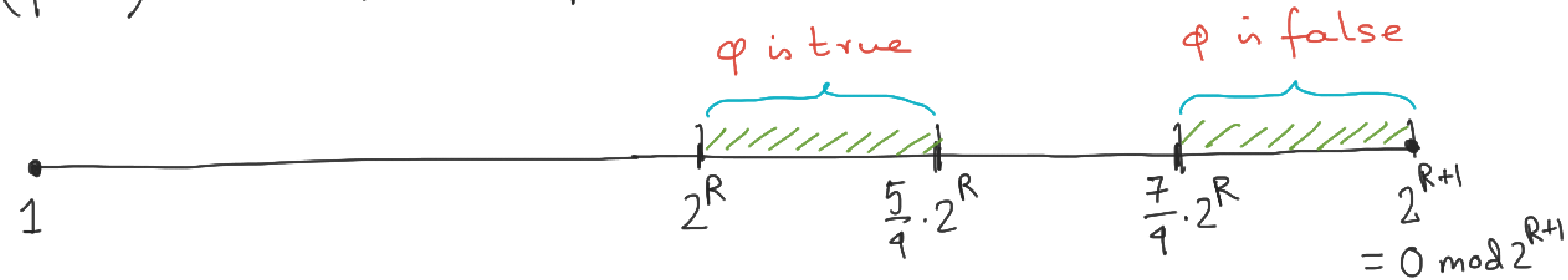
$$\boxed{S := \sum_{\underline{r} \in \{0,1\}^R} \#\left(h(g(\varphi, \underline{r}))\right).} \quad (3)$$

- Obs: (a) If φ is true, then at least $\frac{3}{4}$ of the summands are $0 \bmod 2^{R+1}$, and the remaining are $-1 \bmod 2^{R+1}$.
Hence, in this case, the above sum lies between -2^R and $-\left(\frac{3}{4} \cdot 2^R\right)$ modulo 2^{R+1}.

- (b) If φ is false, then at most $\frac{1}{4}$ of the summands are $0 \bmod 2^{R+1}$, and the remaining are $-1 \bmod 2^{R+1}$.
Hence, in this case, the sum lies between $-\left(\frac{1}{4} \cdot 2^R\right)$ and 0 modulo 2^{R+1}.

- **Note:** ▷ $-2^R = 2^R \mod 2^{R+1}$; $-\left(\frac{3}{4}\cdot 2^R\right) = 2^{R+1} - \frac{3}{4}\cdot 2^R = \frac{5}{4}\cdot 2^R \mod 2^{R+1}$.

  ▷ $-\left(\frac{1}{4}\cdot 2^R\right) = 2^{R+1} - \frac{1}{4}2^R = \frac{7}{4}\cdot 2^R \mod 2^{R+1}$.



- So, if we know the sum S (given by Eqn (3)), then we can find out if φ is true or false simply by checking if $S \mod 2^{R+1} \in \left[2^R, \frac{5}{4}2^R\right]$ or

$$S \mod 2^{R+1} \in \left[\frac{7}{4}2^R, 2^{R+1}\right].$$

- $$S = \sum_{\underline{r} \in \{0,1\}^R} \#\Big(h(g(\varphi, \underline{r}))\Big) = \sum_{\underline{r} \in \{0,1\}^R} \sum_{\underline{\omega} \in \{0,1\}^{|\underline{\omega}|}} h(g(\varphi, \underline{r}))(\underline{\omega}). \qquad \text{—— (4)}$$

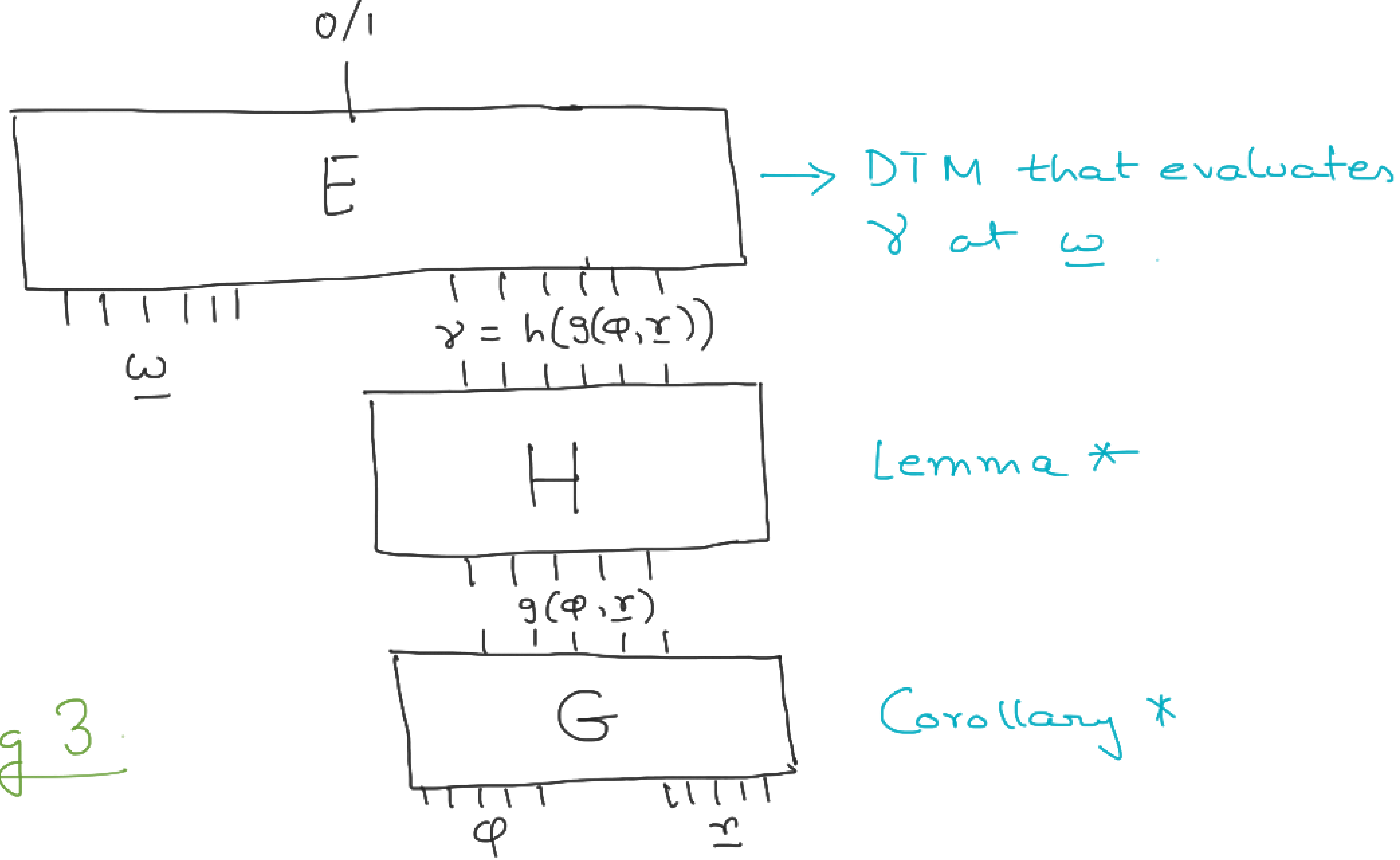


Fig 3

- By fixing φ, we can view the above computation as a DTM M_φ on inputs $\underline{r}$ & $\underline{\omega}$.
- By the Cook-Levin theorem, there's a poly-size ckt Γ_φ that captures the computation of M_φ, i.e.,

$$\Gamma_\varphi(\underline{r}, \underline{\omega}) = M_\varphi(\underline{r}, \underline{\omega}) \quad \forall\ \underline{r}, \underline{\omega}.$$

- **Remark:** Γ_φ is <u>poly-time computable</u> from φ.

- From Eqn (4),

$$S = \sum_{\underline{r} \in \{0,1\}^R} \sum_{\underline{\omega} \in \{0,1\}^{|\underline{\omega}|}} M_\varphi(\underline{r}, \underline{\omega}) = \sum_{\underline{r}} \sum_{\underline{\omega}} \Gamma_\varphi(\underline{r}, \underline{\omega})$$

$$= \# \Gamma_\varphi(\underline{r}, \underline{\omega}).$$

- Therefore, by querying Γ_φ to the #SAT oracle, we can find out if φ is true/false. $\Longrightarrow PH \subseteq P^{\#SAT}$.
- Only <u>one</u> query to the #SAT oracle is required.