Computational Complexity Theory

Lecture 7: Time Hierarchy Theorem; Ladner's Theorem

Department of Computer Science,
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Recap: Nondeterministic Turing Machines

- A *nondeterministic Turing machine* is like a deterministic Turing machines but with two transition functions.
- It is formally defined by a tuple $(\Gamma, Q, \delta_0, \delta_1)$. It has a special state q_{accept} in addition to q_{start} and q_{halt}.
- At every step of computation, the machine applies one of two functions δ_0 and δ_1 *arbitrarily*.
- Unlike DTMs, NTMs are **not intended to be physically realizable** (because of the arbitrary nature of application of the transition functions).

Recap: Nondeterministic Turing Machines

- Definition. An NTM M *accepts* a string $x \in \{0,1\}^*$ iff on input x there ***exists*** a sequence of applications of the transition functions δ_0 and δ_1 (beginning from the start configuration) that makes M reach q_{accept}.

- Defintion. An NTM M *decides* L in $T(|x|)$ time if
  - M accepts x ⟷ $x \in L$
  - On every sequence of applications of the transition functions on input x, M either reaches q_{accept} or q_{halt} within $T(|x|)$ steps of computation.

Recap: Alternate characterization of NP

- **Definition.** A language L is in $NTIME(T(n))$ if there's an NTM M that decides L in $c.\ T(n)$ time on inputs of length n, where c is a constant.

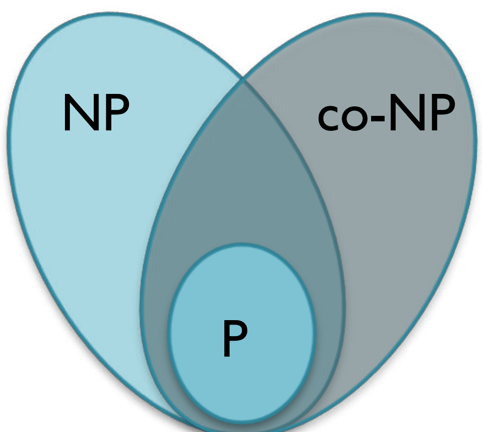
- **Theorem.** $NP = \bigcup_{c > 0} NTIME(n^c)$.

Recap: Class co-NP

- Definition. For every $L \subseteq \{0,1\}^*$ let $\overline{L} = \{0,1\}^* \setminus L$.

  A language L is in co-NP if $\overline{L}$ is in NP.

- Example. $\overline{\text{SAT}} = \{\phi : \phi$ is *not* satisfiable$\}$.



Recap: Alternate definition of co-NP

- Recall, a language $L \subseteq \{0,1\}^*$ is in NP if there's a *poly function* p and a *poly-time verifier* M such that

$$x \in L \Leftrightarrow \exists u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x,u) = 1$$

$$x \in \overline{L} \Leftrightarrow \forall u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x,u) = 0$$

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- **Definition.** A language $L \subseteq \{0,1\}^*$ is in co-NP if there's a *poly function* p and a *poly-time TM* M such that

$$x \in L \Leftrightarrow \forall u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x,u) = 1$$

for NP this was $\exists$

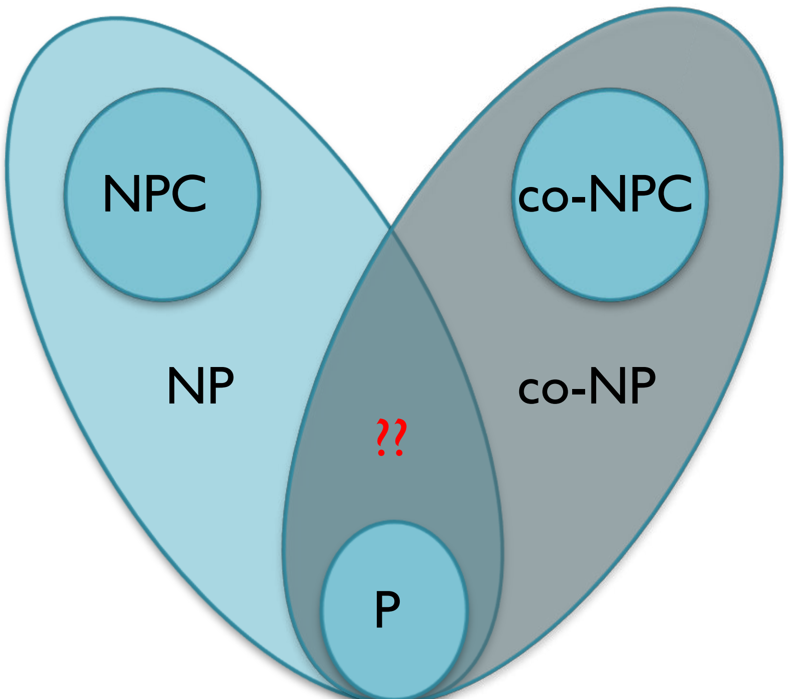
Recap: co-NP-completeness

- Definition. A language L’ $\subseteq \{0,1\}^*$ is co-NP-complete if
  - L’ is in co-NP
  - Every language L in co-NP is polynomial-time (Karp) reducible to L’.

- Theorem. $\overline{\text{SAT}}$ and TAUTOLOGY are co-NP-complete.

- Theorem. If L in NP-complete then L is co-NP-complete

Recap: co-NPC not in NP



If a co-NP-complete language belongs to NP then

co-NP $\subseteq$ NP

→ co-NP = NP

Let C_1 and C_2 be two complexity classes.

If $C_1 \subseteq C_2$, then co-$C_1 \subseteq$ co-C_2.

Obs. co-(co-C) = C .

Recap: Integer factoring in NP∩co-NP

- Integer factoring.

  FACT = {(N, U): there's a prime in [U] dividing N}

- Claim. FACT ∈ NP ∩ co-NP

- So, FACT is NP-complete implies NP = co-NP.

- FACT not known to be in P.

Recap: Class EXP

- Definition. Class EXP is the exponential time analogue of class P.

$$EXP = \bigcup_{c \geq 1} DTIME(2^{n^c})$$

- Observation. $P \subseteq NP \subseteq EXP$

- ***Exponential Time Hypothesis.*** *(Impagliazzo & Paturi 1999)* Any algorithm for 3-SAT takes $\geq 2^{\delta . n}$ time, where $\delta > 0$ is some fixed constant and n is the no. of variables.

ETH ➡ $P \neq NP$

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Is $P \subsetneq EXP$?

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     If M_α takes T time on x then U takes $O(T \log T)$ time to simulate M_α on x.

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1. There's a universal TM U that when given strings α and x, simulates M_α on x with only a *small* overhead.

2. Every string represents some TM, and every TM can be represented by *infinitely many* strings.

Time Hierarchy Theorem

- An application of Diagonalization

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be <u>time-constructible</u> functions s.t.,

  $$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. *(Hartmanis & Stearns 1965)*

  $$\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$$

- Theorem. $\mathrm{P} \subsetneq \mathrm{EXP}$

- This type of results are called **<u>lower bounds</u>**.

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  Task: Show that there's a language L decided by a TM D with time complexity $O(n^2)$ s.t., any TM M with runtime $O(n)$ cannot decide L.

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D's time steps not M_x's time steps.

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     a. If M_x stops and outputs b then output 1-b.

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D outputs the **opposite** of what M_x outputs.

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  D runs in $O(n^2)$ time as n^2 is time-constructible.

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  Claim. There's no TM M with running time $O(n)$ that decides L (the language accepted by D).

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Contradiction! M does not decide L.

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- Theorem. $P \subsetneq EXP$

  Proof. Similar (homework)

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- Conjecture. ***No*** algorithm solves 3SUM in $O(n^{2-\varepsilon})$ time for some constant $\varepsilon > 0$.

- However, there's a $\sim O(n^2 / (\log n)^2)$ time algorithm for 3SUM. ("~" suppressing a poly(log log n) factor.)

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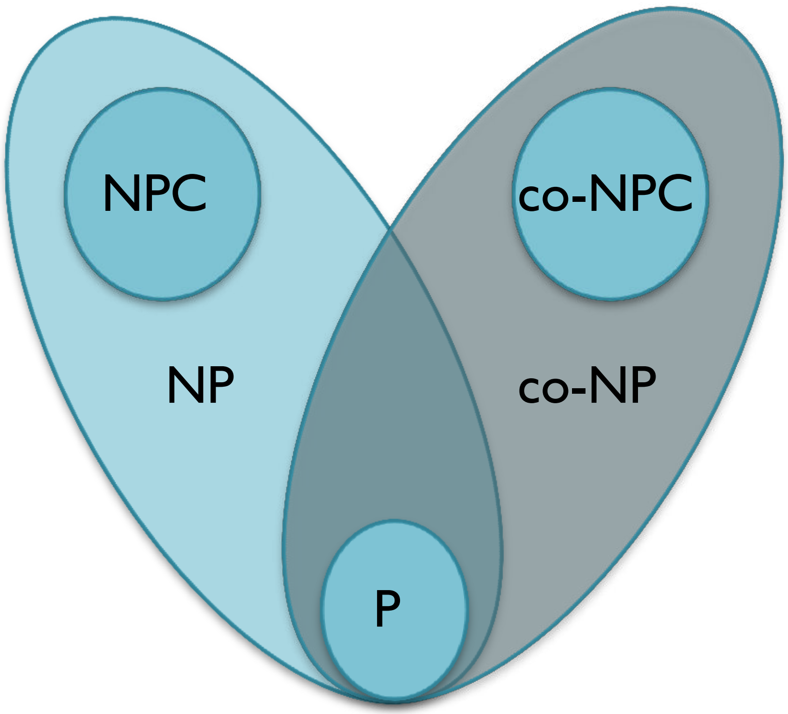
- Theorem *(Patrascu & Williams 2010)*. ETH implies kSUM requires $n^{\Omega(k)}$ time.

Revisiting NP∩co-NP



Conjecture: NP ≠ co-NP

⇓

P ≠ NP

General belief: P ≠ NP ∩ co-NP

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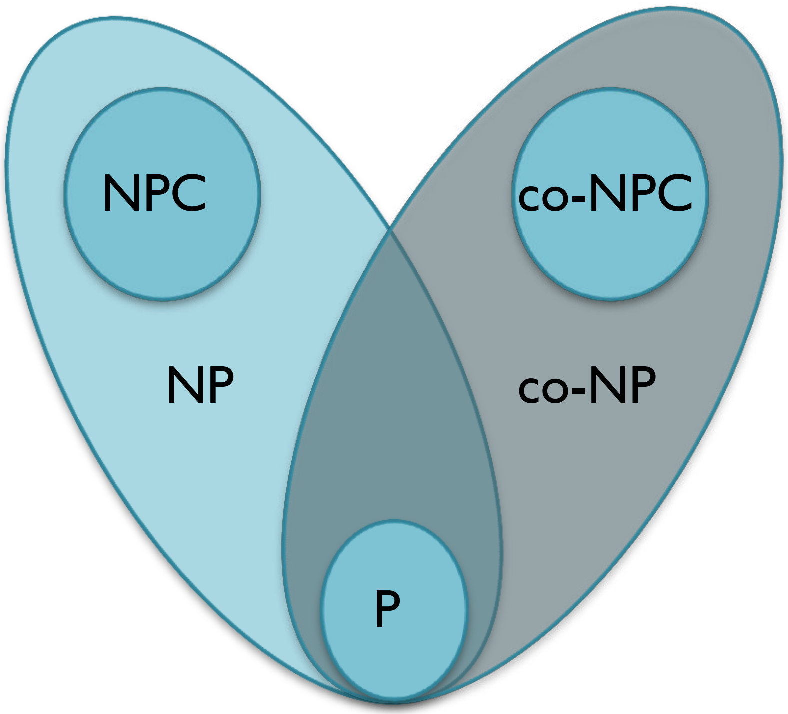

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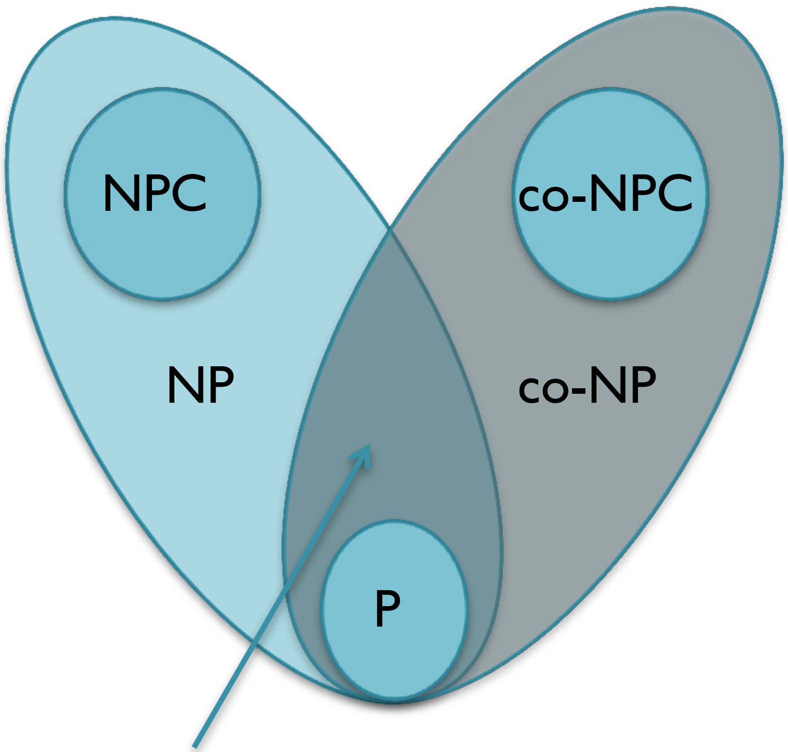
Edmonds (1966)

...conjectured P = NP ∩ co-NP

Revisiting NP∩co-NP



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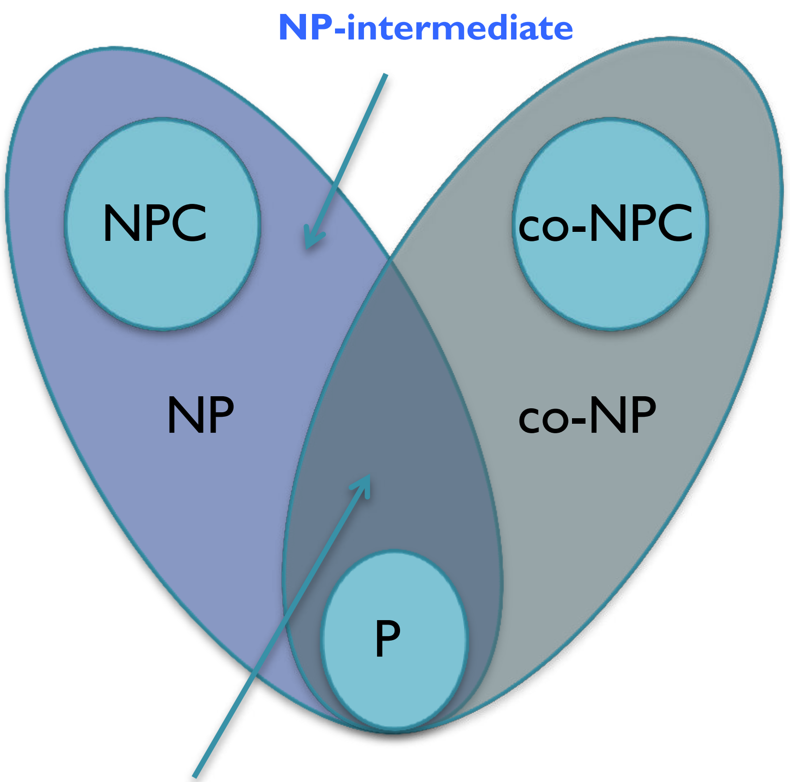
Check:
https://cstheory.stackexchange.com/questions/20021/reasons-to-believe-p-ne-np-cap-conp-or-not

- Integer factoring (FACT)
- Approximate shortest vector in a lattice

Ref: "*Lattice problems in NP∩co-NP*" by Aharonov & Regev (2005)

NP-intermediate problems



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Obs: If NP ≠ co-NP and FACT ∉ P then FACT is NP-intermediate.

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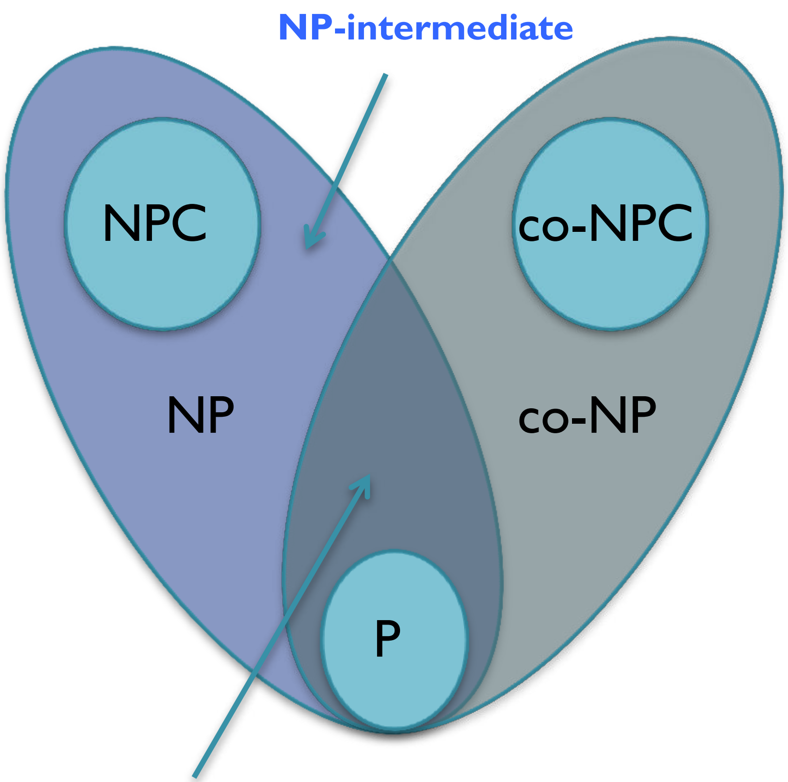

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Ladner's theorem: P ≠ NP implies existence of a NP-intermediate language.

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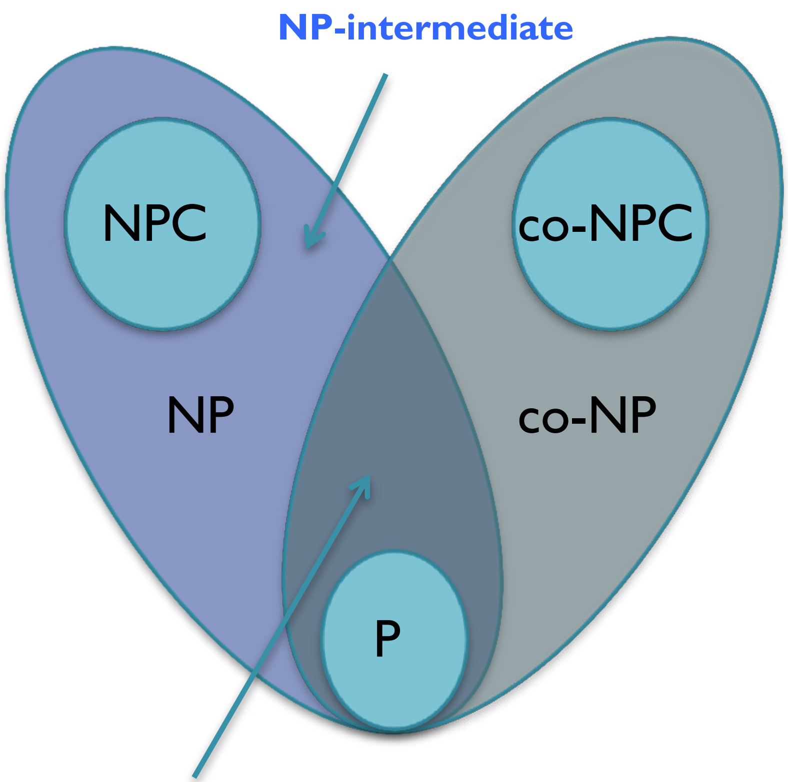

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(proved using ***diagonalization***)

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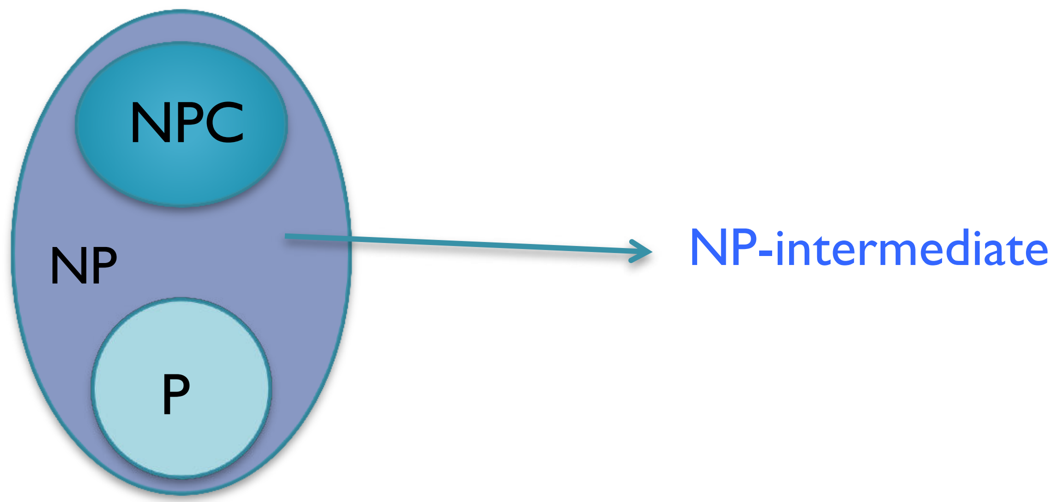
Ladner’s Theorem

- Another application of Diagonalization

NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.



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- Theorem. *(Ladner 1975)* If $P \neq NP$ then there is a NP-intermediate language.

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  Proof. A delicate argument using diagonalization.

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  Proof. Let H: $\mathbb{N} \rightarrow \mathbb{N}$ be a function.

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- Theorem. *(Ladner 1975)* If $P \neq NP$ then there is a NP-intermediate language.

  Proof. Let $H: \mathbb{N} \to \mathbb{N}$ be a function.

  Let $SAT_H = \{\Psi 0 1^{m^{H(m)}} : \Psi \in SAT \text{ and } |\Psi| = m\}$

NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.

- Theorem. *(Ladner 1975)* If $P \neq NP$ then there is a NP-intermediate language.

  Proof. Let $H: \mathbb{N} \rightarrow \mathbb{N}$ be a function.

  Let $SAT_H = \{\Psi 0 1^{m^{H(m)}} : \Psi \in SAT \text{ and } |\Psi| = m\}$

  H would be defined in such a way that SAT_H is NP-intermediate (assuming $P \neq NP$)

Ladner's theorem: Constructing H

- Theorem. There's a function $H: \mathbb{N} \rightarrow \mathbb{N}$ such that

   1. $H(m)$ is computable from m in $O(m^3)$ time.

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Proof: Later (uses diagonalization).

Let's see the proof of Ladner's theorem assuming the existence of such a "special" H.

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  - Check if $\phi\, 0\, 1^{m^{H(m)}}$ belongs to SAT_H .
- As $P \neq NP$, it must be that $SAT_H \notin P$.

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- Suppose SAT_H is NP-complete. Then $H(m) \to \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad\qquad \phi \xmapsto{f} \Psi\, 0\, 1^k$$

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Either $m \leq m_0$ (in which case the task reduces to checking if a constant-size Ψ is satisfiable),

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- On input ϕ, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.

or $H(m) > 2c$ (as $H(m)$ tends to infinity with m).

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- On input ϕ, compute $f(\phi) = \Psi\ 0\ 1^k$. Let $m = |\Psi|$.
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- Hence, w.l.o.g. $\quad |f(\phi)| \geq k > m^{2c}$

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- On input ϕ, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.
- Hence, $\sqrt{n} \geq m$.

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- On input ϕ, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
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Thus, checking if an n-size formula ϕ is satisfiable reduces to checking if a $\sqrt{n}$-size formula Ψ is satisfiable.

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Do this recursively! Only $O(\log \log n)$ recursive steps required.

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  - Compute $H(m)$ and check if $k = m^{H(m)}$.
  - Hence, $\sqrt{n} \geq m$. Also $\phi \in SAT$ iff $\Psi \in SAT$

- Hence SAT_H is not NP-complete, as $P \neq NP$.

Natural NP-intermediate problems ??

- Integer factoring
- Approximate shortest vector in a lattice
- Minimum Circuit Size Problem

  ("*Multi-output MCSP is NP-hard*", Ilango, Loff & Oliveira 2020)

- Graph isomorphism

  ("*GI in QuasiP time*", Babai 2015)