Parallel Algorithms for Perfect Matchings

Prathik, Prayas & Sreenivas

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Matching is as Easy as Matrix Inversion

Brief Overview

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- The core feature of this algorithm is that the only non-trivial computation involved is the inversion of an integer matrix
- Since there are existing parallel algorithms for matrix inversion, this shows that maximum matching is in RNC^2
- As this algorithm is designed with parallel computation in mind, the core problem we must solve is that of coordinating all the parallel processors to seek the same solution
- This is done by using the isolating lemma, which can (with a high probability) single out one matching in the graph

The Isolating Lemma

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- This provides a relation between the parallel complexity of an arbitrary search problem and the corresponding weighted decision problem
- In particular, we show that Exact Matching (a problem not known to be in P) is in RNC^2
- The isolating lemma can also be used to provide a simpler proof of the Valiant-Vazirani theorem, which shows that SAT remains hard (under randomized reductions) even if it is known that there is at most one satisfying assignment

History

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History

- The maximum matching problem is a natural and well-studied problem, with the very idea of considering "tractable" to mean "poly-time solvable" first arising in the context of solving the general matching problem
- Parallel algorithms to solve the problem typically require a different, often algebraic or probabilistic, method from solving them sequentially
- A critical idea used is Tutte's theorem (1947), which states that a graph has a perfect matching iff a certain "Tutte" matrix of indeterminates is non-singular

First Attempts

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This Paper

This algorithm directly finds a perfect matching, and is faster (RNC^2) while using $O(n^{3.5}m)$ processors

The Isolating Lemma

Definition

A *set system* (S, F) consists of a finite set S of elements, $S = \{x_1, ..., x_n\}$ and a family F of subsets of S, i.e., $F = \{S_1, ..., S_k\}$, where $S_i \subseteq S$ for $1 \leq i \leq k$

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We can assign a weight w_i to each element $x_i \in S$, and define the weight of a subset S_j to be $\sum_{x_i \in S_j} w_i$

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Lemma

Let (S, F) be a set system with weights assigned uniformly and independently at random from $[1, 2n]$. Then:

$$\mathbb{P}[\textit{there is a unique minimum weight set in } F] \geq \frac{1}{2}$$

Proof

- Say we fix the weights for all elements except some x_i. Then define the *threshold* for x_i to be the real number (possibly negative) α_i such that if $w_i \leq \alpha_i$, then x_i is in some minimum weight subset, while if $w_i > \alpha_i$, then x_i is not in any minimum weight subset

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- Note that if $w_i < \alpha_i$, then x_i must be in *every* minimum weight subset – any ambiguity about x_i only occurs if $w_i = \alpha_i$, since then there is some minimum weight subset containing x_i and some other minimum weight subset not containing x_i

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- We will say that an element x_i is *singular* if $w_i = \alpha_i$
- Note that there can only be multiple minimum weight subsets if there is at least one singular element

Key Observation

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$$\mathbb{P}[x_i \text{ is singular, i.e., } w_i = \alpha_i] \leq \frac{1}{2n}$$

- Then by the union bound over the n elements, we have:

$$\mathbb{P}[\text{at least one element is singular}] \leq \frac{1}{2}$$

- This completes the proof of the lemma by the previous observation

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- We design our parallel algorithm to seek this perfect matching
- Let D be the biadjacency matrix of G and generate B from D by replacing every $d_{ij} = 1$ with $b_{ij} = 2^{w_{ij}}$, where w_{ij} is the weight for the edge (u_i, v_j)

The Matching Algorithm - Bipartite Graphs

Lemma

Say the minimum weight perfect matching is unique. Call it M and let its weight be w. Then $|B| \neq 0$ and moreover the highest power of 2 dividing $|B|$ is 2^w

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- First, note that any perfect matching corresponds to some permutation $\sigma \in S_n$, so we can define the value of a perfect matching by the following:

$$value(\sigma) = \prod_{i=1}^{n} b_{i\sigma(i)}$$

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$$value(\sigma) = \prod_{i=1}^{n} b_{i\sigma(i)}$$

- Then note that $value(\sigma) \neq 0$ iff σ corresponds to a perfect matching (since σ corresponds to a perfect matching iff $(u_i, v_{\sigma(i)})$ is an edge for $1 \leq i \leq n$)

Proof of the Lemma

- Now note the well-known expansion for the determinant,

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- For any other matching, the corresponding weight $w' > w$, so that the permutation corresponding to it has a value that is a higher power of 2
- Thus, 2^w must be the highest power of 2 that divides $|B|$

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- If $(u_i, v_j) \in M$, then note that one permutation in the above, σ_M, has value 2^w, and everything else has a value that is 0 or a higher power of 2, so $|B_{ij}|2^{w_{ij}}$ is an odd multiple of 2^w

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- On the other hand, if $(u_i, v_j) \notin M$, then every permutation above must have a value that is 0 or a higher power of 2, and thus 2^{w+1} must divide $|B_{ij}|2^{w_{ij}}$

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- This algorithm will find a perfect matching (if it exists) with a probability of at least $1/2$ (note that checking if the output is a perfect matching is trivial)
- The only non-trivial step here is the computation of the determinant and adjoint of an integer matrix, which is equivalent to finding the inverse of the matrix

Tutte's Theorem

Generalizing this algorithm to general graphs is not very difficult; it only requires us to work with the Tutte matrix instead

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Definition

Note that given a graph $G = (V, E)$, its *adjacency matrix* is an $n \times n$ symmetric matrix D with $d_{ij} = 1$ if $(v_i, v_j) \in E$, and $d_{ij} = 0$ otherwise. The *Tutte matrix* is then a skew-symmetric matrix A obtained from D as follows: if $d_{ij} = d_{ji} = 1$, with $j < i$, then replace them with x_{ij} and $-x_{ij}$, so that entries above the diagonal are positive, while leaving 0s unchanged.

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Theorem

Let $G = (V, E)$ be a graph with Tutte matrix A. Then $|A| \neq 0$ iff G has a perfect matching

Generalizing the Algorithm

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- However, these proofs (which involve looking at Tutte matrices) have been skipped here for the sake of brevity
- The only non-trivial step is equivalent to matrix inversion, for which Pan's randomized algorithm can be used which uses $O(n^{3.5}m)$ processors and $O(\log^2 n)$ time to invert a $n \times n$ matrix with m-bit integer entries

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- Then we use a generalization of the isolating lemma by assigning a weight of $mnw(e) + r(e)$, where $r(e)$ is chosen uniformly and independently at random from $[1, 2m]$
- We can then apply the algorithm stated above, requiring $O(n^{3.5}mW)$ processors, where W is the weight of the heaviest edge - so this problem is in RNC^2 as well if the weights are given in unary

Vertex Weighted Matching

- Note that the problem of finding a maximum matching is easily converted to one of finding a minimum weight perfect matching by extending the graph to a complete graph and providing a weight of 1 all new edges and 0 to existing edges

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- Define $V' \subseteq V$ to be a matching set if V' is the set of vertices matched by any maximum matching
- The problem then reduces to one of finding the heaviest matching set, along with finding a perfect matching in the set of vertices within the matching set

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- Say this were not the case, so L (the lexicographically largest matching set) and H (the heaviest matching set) differ at some point, the earliest of which is, say, u
- Clearly u is matched in L but not H, so consider the symmetric difference of L and H, which must have an alternating even length path to some vertex v

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- The symmetric difference of H with this path then contains u instead of v and remains a matching, but this new matching must be heavier as u is heavier than v - a contradiction

Vertex Weighted Matching

- The symmetric difference of H with this path then contains u instead of v and remains a matching, but this new matching must be heavier as u is heavier than v - a contradiction
- Finding the lexicographically largest matching set is known to be in RNC^2, and since we have shown that finding a perfect matching within this matching set is also in RNC^2, it follows that the vertex weighted matching problem is also in RNC^2

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- Many search problems are reducible to their corresponding decision problem via self-reducibility, but this reduction is not easy to paralellize
- Self-reduction typically leads to the lexicographically first solution, and for many problems finding such a solution is known to be P-complete in spite of efficient parallel algorithms existing for the unrestricted search problem
- This was first studied by Karp, Upfal and Wigdersion, who gave an RNC^2 procedure for the search problem by using oracle access to the ranking function

Parallel Complexity of Search vs Decision Problems

- Many search problems are reducible to their corresponding decision problem via self-reducibility, but this reduction is not easy to paralellize
- Self-reduction typically leads to the lexicographically first solution, and for many problems finding such a solution is known to be P-complete in spite of efficient parallel algorithms existing for the unrestricted search problem
- This was first studied by Karp, Upfal and Wigderson, who gave an RNC^2 procedure for the search problem by using oracle access to the ranking function
- Here, we reduce the general search problem to a *weighted decision problem*, with polynomially bounded weights

Parallel Complexity of Search vs Decision Problems

Theorem

Let (S, F) be a set system and O be an oracle for the weighted decision problem of checking if there a set in F with a weight less than k when every element of S is assigned a polynomially bounded weight. Then there is an RNC^1 procedure using the oracle O to find a set in F.

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- First, we find the minimum weight for any set in F via binary search, which takes $O(\log n)$ calls to O
- We can then determine if some element is in the minimum weight set by increasing its weight and checking if the minimum weight set increased in weight (with one call to O)

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- Using this, we find an RNC^2 procedure for the **exact matching** problem, which is not known to be solvable in deterministic polynomial time
- In the **exact matching** problem, a graph $G = (V, E)$ is given, with a subset $E' \subseteq E$ of **red edges**, and the goal is to find a perfect matching with exactly k red edges, for some positive integer k

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- The resulting matrix, say B, is then skew symmetric, so $|B| = (pf(B))^2$, where $pf(B)$ represents the Pfaffian of B

Exact Matching

- Note that the Pfaffian is given by the formula:

$$pf(B) = \frac{1}{2^{\frac{n}{2}}\frac{n}{2}!}\sum_{\sigma\in S_n} sign(\sigma)\prod_{i=1}^{\frac{n}{2}} a_{\sigma(2i-1)\sigma(2i)}$$

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- This yields an RNC^2 procedure for the exact matching problem

Valiant Vazirani Theorem

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- CLIQUE is the problem if finding if there is a clique of size k given a graph $G = (V, E)$, while UCLIQUE is the same with the knowledge that there is at most one such clique
- UCLIQUE is similarly related to USAT

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The reduction from CLIQUE to UCLIQUE is as follows:

- First, we assign random weights $w(v)$ to every vertex $v \in V$ uniformly and independently from $[1, 2n]$, so that by the isolating lemma, the maximum weight clique (of size k) will be unique with a probability of at least $1/2$

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- Finally, choose a random integer $r \in [1, 2nk]$, and let $k' = 2nk^2 + r$, giving us the transformed problem (G', k')

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- Then the following will hold:
  1. $(G, k) \notin \text{CLIQUE} \implies (G', k') \notin \text{UCLIQUE}$
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- With a probability of at least $1/2$, the maximum weight clique (of size k) is unique, with a weight w^* (say). Then exactly one clique in G' can have a size of $2nk^2 + w^*$, since cliques in G' corresponding to sizes other than k are transformed into cliques of sizes less than $2nk^2$ or more than $2nk(k+1)$, and only one clique in G has weight w^*

Discussion

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- Finally, it is as yet unknown if the perfect matching problem is in NC

Bipartite Perfect Matching is in Quasi-NC

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There are NC algorithms for special classes of graphs:

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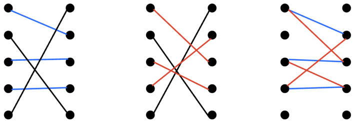

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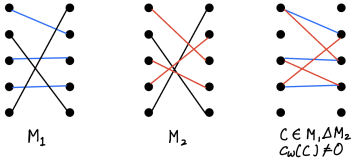

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- Subgraph $G' \subseteq G$ which is the union of minimum weight PMs in G. In the bipartite case, graph G' is significantly smaller than G.
- We show that G' does not contain any cycle with a nonzero circulation. This means that G' does not contain any small cycles.

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Next, we show that for a graph which has no cycles of length $< r$, the number of cycles of length $< 2r$ is polynomially bounded. This motivates the following strategy which works in $\log n$ rounds:

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- Since the graph obtained after $(i-1)-$ th rounds has no cycles of length 2^{i-1}, the number of cycles of length 2^i is small.
- In $\log n$ rounds, we get a unique minimum weight perfect matching.

An *RNC* Algorithm for SEARCH-*PM*

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Natural extension of weight function w to $\mathbb{R}^m$:

$$w(\mathbf{x}) = \sum_{e \in E} w(e) x_e$$

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For any matching M, we have $w(M) = w(\mathbf{x}^M)$

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Let M^* be a PM in G of minimum weight. Then,

$$w(M^*) = \min\{w(\mathbf{x}) | \mathbf{x} \in PM(G)\}$$

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Lemma

G be a bipartite graph & $\mathbf{x} \in \mathbb{R}^m$. $\mathbf{x} \in PM(G)$ if and only if

$$\sum_{e \in \delta(v)} x_e = 1 \quad v \in V, \tag{1}$$

$$x_e \geq 0 \quad e \in E, \tag{2}$$

where $\delta(v)$ denotes the set of edges incident on the vertex v.

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$G(V, E)$ **has a PM**.

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Cycle C in G is a *nice cycle*, if the subgraph $G - C$ still has a PM.

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Nice cycle can be obtained from symmetric difference of 2 PMs.

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Note that a nice cycle is always an even cycle.

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Circulation $c_w(C)$

For an even length cycle $C = (v_1, v_2, \cdots, v_k)$

$$c_w(C) := |w(v_1, v_2) - w(v_2, v_3) + w(v_3, v_4) - \cdots - w(v_k, v_1)|$$

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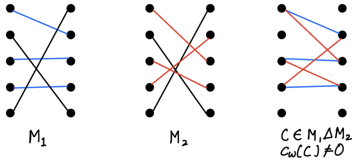

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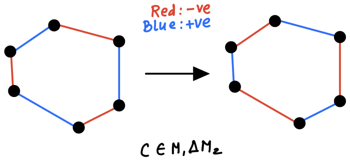

Forcing Nonzero Circulation

Lemma

([CRS95]).For any $s \in \mathbb{N}$ one can construct a set of $O(n^2 s)$ weight assignments with weights bounded by $O(n^2 s)$, such that for any set of s cycles, one of the weight assignments gives nonzero circulation to each of the s cycles. link

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Let $\{e_i\}_{i \in [m]}$ enumerates E. Define, $w(e_i) = 2^{i-1} \forall i \in [m]$.

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Set $lcm(2, 3, \cdots, t) > \prod_{i=1}^{s} c_w(C_i)$. We know $\prod_{i=1}^{s} c_w(C_i) < 2^{n^2 s} \& lcm(2, 3, \cdots, t) > 2^t$ for $t \geq 7$. Choosing $t = n^2 s$ suffices. □

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Let C be a cycle in G such that $c_w(C) \neq 0$. E_1 be the union of all minimum weight PMs in G. $G_1(V, E_1)$ does not contain C.

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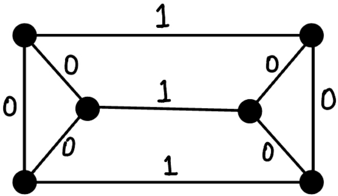

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We show that when we move from point $\mathbf{x}$ along the cycle C, we reach a point in the PM polytope with a weight $< q$. □

Define $\mathbf{y} \in \mathbb{R}^m$:

$$y_e = \begin{cases} x_e + (-1)^i \varepsilon, & \text{if} e = e_i, \text{for some } i \in [p], \\ x_e, \text{otherwise} \end{cases}$$

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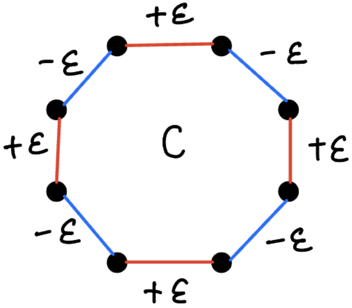

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Now we show that $\mathbf{y} \in PM(G)$.But, as $w(\mathbf{y}) < q$ there must be a corner point of $PM(G)$, corresponding to a PM in G with weight $< q$. This would give us a contradiction, completing the proof.

Lemma

G be a bipartite graph & $\mathbf{x} \in \mathbb{R}^m$. $\mathbf{x} \in PM(G)$ if and only if

$$\sum_{e \in \delta(v)} x_e = 1 \quad v \in V, \tag{3}$$

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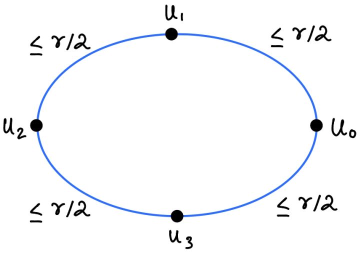
Lemma

Let H be a graph with n nodes that has no cycles of length $\leq r$. Let $r' = 2r$ when r is even, and $r' = 2r - 2$ otherwise. Then H has $\leq n^4$ cycles of length $\leq r'$.



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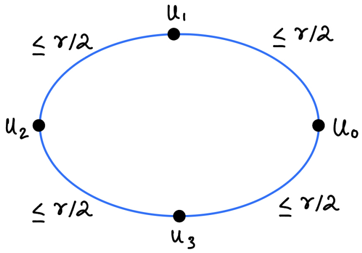

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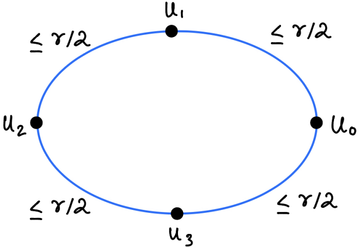

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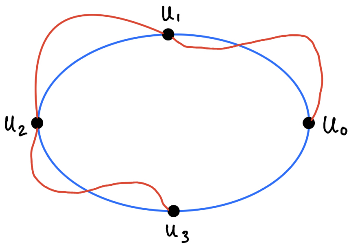

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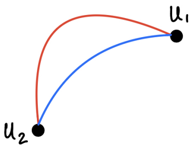
- C' forms a cycle of length at most r. Contradiction.



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$\forall i \in [k]$ let M_1 be PM's in G_i and M_2 be a PM G_{i-1} which is not in G_i. Then $w(M_1) < w(M_2)$.

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In quasi-NC^1, one can construct a set of $O(n^6 \log n)$ integer weight functions on $[n/2] \times [n/2]$, where weights have $O(\log^2 n)$ bits, such that for any bipartite graph with n nodes, one of the weight functions is isolating.

One can decide the existence of a perfect matching in a bipartite graph in quasi-NC_2 as follows:

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- Weights have $O(\log^2 n)$ bits, determinant entries have quasi-polynomial bits.
- As we need to compute $2^{O(\log^2 n)}$-many determinants in parallel, our algorithm is in quasi-NC^2 with circuit size $2^{O(\log^2 n)}$.

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- In this case, the algorithm computes a set of edges M_0 that might or might not be a perfect matching. However, it is easy to verify if M_0 is indeed a perfect matching, and in this case,we will output M_0.

Search Problem

- For a weight function w which is isolating, the algorithm outputs the unique minimum weight PM M. If we have a weight function w_0 which is not isolating, still $det(A)$ might be non-zero with respect to w_0.
- In this case, the algorithm computes a set of edges M_0 that might or might not be a perfect matching. However, it is easy to verify if M_0 is indeed a perfect matching, and in this case,we will output M_0.
- As the algorithm involves computation of similar determinants as in the decision algorithm, it is in quasi-NC_2 with circuit size $2^{O(\log^2 n)}$.

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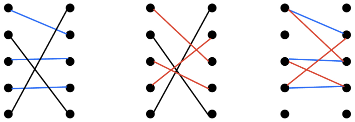

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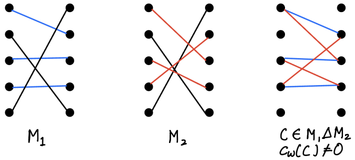

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- Then, we consider the subgraph G' of G which is the union of minimum weight PMs in G. In the bipartite case, graph G' is significantly smaller than the original graph G.
- In particular, we showed that G' does not contain any cycle with a nonzero circulation. This meant that G' does not contain any small cycles.

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Next, we showed that for a graph which has no cycles of length $< r$, the number of cycles of length $< 2r$ is polynomially bounded. This motivated the following strategy which works in $\log n$ rounds:

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- *Instead of quasi-NC, we can get an RNC-circuit but with only* $O(\log^2 n)$ *random bits.*
- *For complete derandomization, it would suffice to bring the number of random bits down to* $O(\log n)$*. Then there are only polynomially many random strings which can all be tested in NC.*

RNC^2 Algorithms with Few Random Bits

- Decision Version
- Search Version

Decision Version

Theorem

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To solve the first problem, we modify Lemma 2.3 to get a random weight assignment which works with high probability. link

The Modified Lemma

Let G be a graph with n nodes and $s \geq 1$.

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Lemma

There is a random weight assignment w which uses $O(\log ns)$ random bits and assigns weights bounded by $O(n^3 s \log ns)$, i.e., with $O(\log ns)$ bits, such that for any set of s cycles, w gives nonzero circulation to each of the s cycles with probability at least $1 - 1/n$.

Proof.

$w(e_i) = 2^{i-1}$.

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Product is bounded by $2^{n^2 s}$, so it has at most $n^2 s$ prime factors. Choose $t = n^3 s$.Then, the random prime works with probability $\geq (1 - 1/n)$.
Now t^{th} prime $\leq 2t \log t = O(n^3 s \log ns)$, by which the weights are bounded by, hence have $O(\log ns)$ bits. □

- We choose each of the weight functions $w_0, w_1, \cdots, w_{k1}$ independently. Probability that all of them provide non-zero circulation on their respective cycles $\geq 1 - k/n \geq 1 - \log n/n$ by union bound.

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$det(A) \neq 0$ if and only if G has a PM.

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Number of random bits:

For a weight assignment w_i, we need $O(\log ns)$ random bits, where $s = n^4$. Number of random bits required for all w_i's together is $O(k \log n) = O(\log^2 n)$. Finally, we need to plug in $O(\log n)$ random bits for each x_i. This again requires $O(\log^2 n)$ random bits.

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In summary, we get an RNC^2-algorithm that uses $O(\log^2 n)$ random bits.

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In NC^2 , we can construct the weight assignments and compute the determinants in each round. As we have $k = O(\log n)$ rounds, the overall complexity becomes NC^3.

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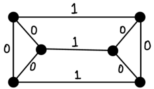

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- Svensson and Tarnawski [ST,17] generalized Quasi-NC^3 for General graphs.