Computational Complexity Theory

Lecture 13: Log-space reductions (contd.); NL-completeness; NL = co-NL

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Recap: NL-completeness

- Recall again, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is L = NL ? …poly-time (Karp) reductions are much too powerful for L.
- We need to define a suitable *'log-space'* reduction.

Recap: Log-space reductions

$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$

- Issue: A log-space TM may not have enough space to write down the whole output $f(x)$ in one shot.
- Solution: Have the log-space TM output a bit of $f(x)$.

- Definition: A function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ is *implicitly log-space computable* if
  1. $|f(x)| \leq |x|^c$ for some constant c,
  2. The following two languages are in L :

$$L_f = \{(x, i) : f(x)_i = 1\} \quad \text{and} \quad L'_f = \{(x, i) : i \leq |f(x)|\}$$

Log-space reductions

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- **Definition:** A language L_1 is *log-space reducible* to a language L_2, denoted $L_1 \leq_l L_2$, if there's an implicitly log-space computable function f such that

$$x \in L_1 \iff f(x) \in L_2$$

Log-space reductions

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- **Solution:** Have the log-space TM output a bit of f(x).

- **Claim:** If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- **Proof:** Let f be the reduction from L_1 to L_2, and g the reduction from L_2 to L_3. We'll show that the function $h(x) = g(f(x))$ is implicitly log-space computable which will suffice as,

$$x \in L_1 \iff f(x) \in L_2 \iff g(f(x)) \in L_3$$

Log-space reductions

$$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$$

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- **Claim:** If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- **Proof:** …Think of the following log-space TM that computes $h(x)_i$ from (x, i). Let
  - ➢ M_f be the log-space TM that computes $f(x)_j$ from (x, j),
  - ➢ M_g be the log-space TM that computes $g(y)_i$ from (y, i).

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- Claim: If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- Proof: …On input x, simulate M_g on $(f(x), i)$ pretending that $f(x)$ is there in some fictitious tape. During the simulation whenever M_g tries to read a j-th bit of $f(x)$, postpone M_g's computation and start simulating M_f on input (x, j).

Log-space reductions

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stores M_g's current configuration

- **Claim:** If $L_1 \le_l L_2$ and $L_2 \le_l L_3$ then $L_1 \le_l L_3$.
- **Proof:** …On input x, simulate M_g on $(f(x), i)$ <u>pretending that</u> $f(x)$ is there in some fictitious tape. During the simulation whenever M_g tries to read a j-th bit of $f(x)$, <u>postpone</u> M_g's computation and start simulating M_f on input (x, j). Space usage $= O(\log |f(x)|) + O(\log |x|)$.

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Log-space reductions

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- Proof: …On input x, simulate M_g on (f(x), i) pretending that f(x) is there in some fictitious tape. During the simulation whenever M_g tries to read a j-th bit of f(x), postpone M_g's computation and start simulating M_f on input (x, j). This shows L_h is in L.

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- Claim: If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- Proof: …Similarly, L'_h is in L and so h is implicitly log-space computable.

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- Solution: Have the log-space TM output a bit of f(x).

- Claim: If $L_1 \leq_l L_2$ and $L_2 \in \mathbf{L}$ then $L_1 \in \mathbf{L}$.
- Proof: Same ideas. (*Homework*)

NL-completeness

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  PATH = {(G,s,t) : G is a digraph having a path from s to t}.

- Theorem: PATH is NL-complete.
- Proof: We've already shown that PATH ∈ NL. Now we'll show that for every L ∈ NL, $L \leq_l$ PATH. We need to come up with an implicitly log-space computable function f s.t.

$$x \in L \iff f(x) \in \text{PATH}$$

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- Proof: (contd.) Let M be a log-space NTM deciding L. Define, $f(x) = (G_{M,x}, C_{start}, C_{accept})$, where $G_{M,x}$ is given as an adjacency matrix. Let $m = O(\log |x|)$ be the no. of bits required to represent a configuration. Then, $|f(x)| = 2^{2m} + 2m = poly(|x|)$.

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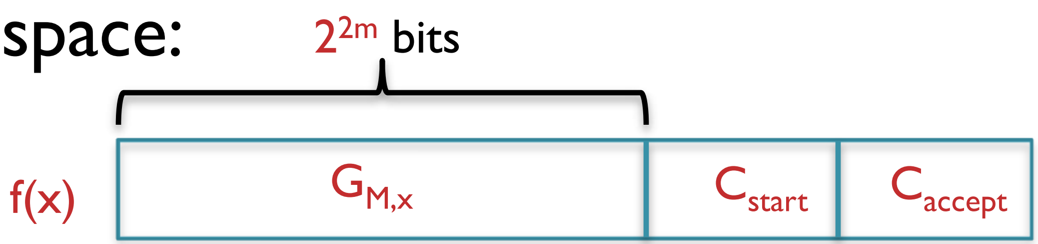
- Proof: (contd.) Let's see how to compute $f(x)_i$ from (x, i) using log-space:



If $i > 2^{2m}$ then i indexes a bit in the (C_{start}, C_{accept}) part of f(x); so $f(x)_i$ can be computed by simply writing down C_{start} and C_{accept}.

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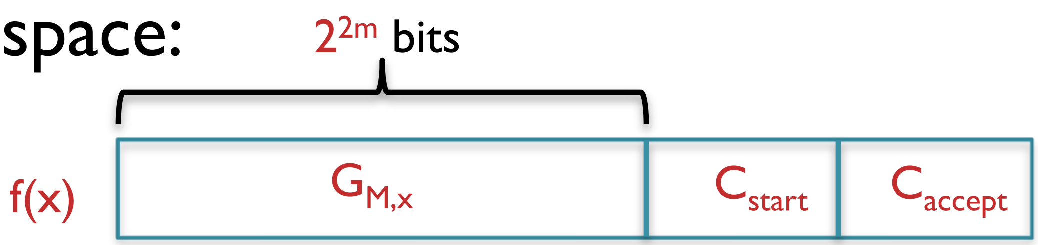

If $i \leq 2^{2m}$ then write i as (C_1,C_2), where C_1 and C_2 are m bits each, and check if C_2 is a neighbor of C_1 in $G_{M,x}$. This takes $O(m)$ space.

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- Theorem: PATH is NL-complete.
- Proof: (contd.) Thus, we've argued that |f(x)| has poly(|x|) length and L_f ∈ L. Similarly, L'_f ∈ L. So, f defines a log-space reduction from L to PATH.

Other NL-complete problems

- Reachability in directed acyclic graphs.
- Checking if a directed graph is strongly connected.
- 2SAT.
- Determining if a word is accepted by a NFA.

An alternate characterization of NL

Certificate definition of NL

- Like NP, it will be useful to have a *certificate-verifier* kind of definition of the class NL.
- We'll see how it helps in proving NL = co-NL i.e., in showing $\overline{\text{PATH}} \in$ NL.

$\overline{\text{PATH}}$ = {(G,s,t): G is a digraph with no path from s to t}

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- Definition.(first attempt) Suppose L is a language, and there's a *log-space verifier* M & a function q s.t.

  $$x \in L \iff \exists u \in \{0,1\}^{q(|x|)} \text{ s.t. } M(x,u) = 1$$

  Should we define q(|x|) as a log function, meaning q(|x|) = O(log |x|) ?

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Should we define $q(|x|)$ as a log function, meaning $q(|x|) = O(\log |x|)$?

…**No, that's too restrictive.** That will imply L ∈ L.

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- Definition.(first attempt) Suppose L is a language, and there's a *log-space verifier* M & a *<u>poly-function</u>* q s.t.

$$x \in L \iff \exists u \in \{0,1\}^{q(|x|)} \text{ s.t. } M(x,u) = 1$$

Is it so that L ∈ NL iff L has such a log-space verifier of the above kind?

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Is it so that L ∈ NL iff L has such a log-space verifier of the above kind?
Unfortunately not!! Exercise: L ∈ NP iff L has such a log-space verifier.

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- Definition.(first attempt) Suppose L is a language, and there's a *log-space verifier* M & a *poly-function* q s.t.

$$x \in L \quad \Longleftrightarrow \quad \exists u \in \{0,1\}^{q(|x|)} \text{ s.t. } M(x,u) = 1$$

Solution: Make the certificate ***<u>read-one</u>*** as described next…

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  $\overline{\text{PATH}}$ = {(G,s,t): G is a digraph with no path from s to t}

- Definition. A tape is called a *read-one tape* if the head moves from left to right and never turns back.

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$\overline{\text{PATH}}$ = {(G,s,t): G is a digraph with <u>no</u> path from s to t}

- Definition. A language L has *read-once certificates* if there's a *log-space verifier* M & a *poly-function* q s.t.

$$x \in L \iff \exists u \in \{0,1\}^{q(|x|)} \text{ s.t. } M(x,u) = 1,$$

where <u>u is given on a read-once input tape</u> of M.

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- Theorem. L ∈ NL iff L has read-once certificates.

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- Theorem. L ∈ NL iff L has read-once certificates.
- Proof. Suppose L ∈ NL. Let N be an NTM that decides L. Think of a verifier M that on input (x, u) simulates N on input x by using u as the nondeterministic choices of N. Clearly |u| = poly(|x|)...

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- Theorem. L $\in$ NL iff L has read-once certificates.
- Proof. (contd.) …as $G_{N,x}$ has poly(|x|) configurations. M scans u from left to right without moving its head backward. So, u is a read-once certificate satisfying,

$$x \in L \quad \Longleftrightarrow \quad \exists u \in \{0,1\}^{poly(|x|)} \text{ s.t. } M(x,u) = 1$$

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- Theorem. L ∈ NL iff L has read-once certificates.
- Proof. (contd.) Suppose L has read-once certificates, and M be a log-space verifier s.t.

$$x \in L \iff \exists u \in \{0,1\}^{q(|x|)} \text{ s.t. } M(x,u) = 1.$$

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- Theorem. L ∈ NL iff L has read-once certificates.
- Proof. (contd.) Now, think of an NTM N that on input x starts simulating M. It guesses the bits of u as and when required during the simulation. As u is read-once for M, there's no need for N to store u.

Certificate definition of NL

- Like NP, it will be useful to have a *certificate-verifier* kind of definition of the class NL.
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- Theorem. L ∈ NL iff L has read-once certificates.
- Proof. (contd.) So, N is a log-space NTM deciding L.

NL = co-NL

Class co-NL

- Definition. A language L is in co-NL if $\overline{L} \in$ NL. L is co-NL-complete if $L \in$ co-NL and for every $L' \in$ co-NL, L' is log-space reducible to L.

  $\overline{\text{PATH}}$ = {(G,s,t): G is a digraph with no path from s to t}
- Obs. $\overline{\text{PATH}}$ is co-NL-complete under log-space reduction.

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- Obs. $\overline{\text{PATH}}$ is co-NL-complete under log-space reduction.

- Obs. If a language L' log-space reduces to a language in NL then $L' \in$ NL. (*Homework*) So, if $\overline{\text{PATH}} \in$ NL then NL = co-NL.

NL = co-NL

- Theorem. (*Immerman-Szelepcsenyi 1987*) $\overline{\text{PATH}} \in \text{NL}$.

NL = co-NL

- Theorem. (*Immerman-Szelepcsenyi 1987*) $\overline{\text{PATH}} \in$ NL.
- Proof. It is sufficient to show that there's a *log-space verifier* M & a *poly-function* q s.t.

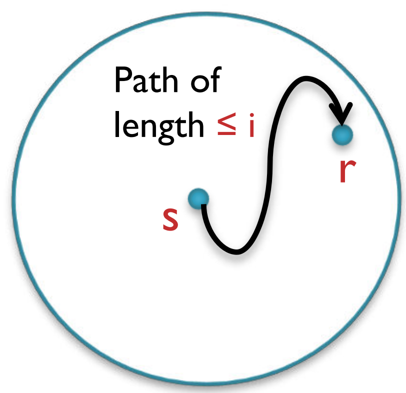
  $x \in \overline{\text{PATH}} \iff \exists u \in \{0,1\}^{q(|x|)}$ s.t. $M(x,u) = 1$,

  where u is given on a read-once input tape of M.

- Let us focus on forming a read-once certificate u that convinces a verifier that there's no path from s to t…

NL = co-NL

- Theorem. (*Immerman-Szelepcsenyi 1987*) $\overline{\text{PATH}} \in$ NL.
- Proof. x = (G,s,t). Let m be the number of nodes in G. Let k_i = no. of nodes reachable from s by a path of length at most i in G.



k_i nodes

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Read-once certificate u is of the form $(u_1, u_2, \ldots, u_m, v)$, where u_i's and v are strings s.t.

(1) reading until $(u_1, u_2, \ldots u_i)$ in a read-once fashion, M knows correctly the value of k_i.

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(1) reading until $(u_1, u_2, \ldots u_i)$ in a read-once fashion, M knows correctly the value of k_i. So, after reading $(u_1, u_2, \ldots u_m)$, M knows k_m, the number of nodes reachable from s.

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(1) reading until $(u_1, u_2, \ldots u_i)$ in a read-once fashion, M knows correctly the value of k_i. So, after reading $(u_1, u_2, \ldots u_m)$, M knows k_m, the number of nodes reachable from s.

(2) v then convinces M (which <u>already knows</u> k_m) that t is not one of the k_m vertices reachable from s.

NL = co-NL

- Theorem. (*Immerman-Szelepcsenyi 1987*) $\overline{\text{PATH}} \in \text{NL}$.
- Proof. We'll design u_i assuming that $u_1, \ldots, u_{i-1}$ have already been constructed and M knows k_{i-1}. Let $r_1, \ldots r_m$ be the nodes of G s.t. $r_1 < r_2 < \ldots < r_m$. Then,

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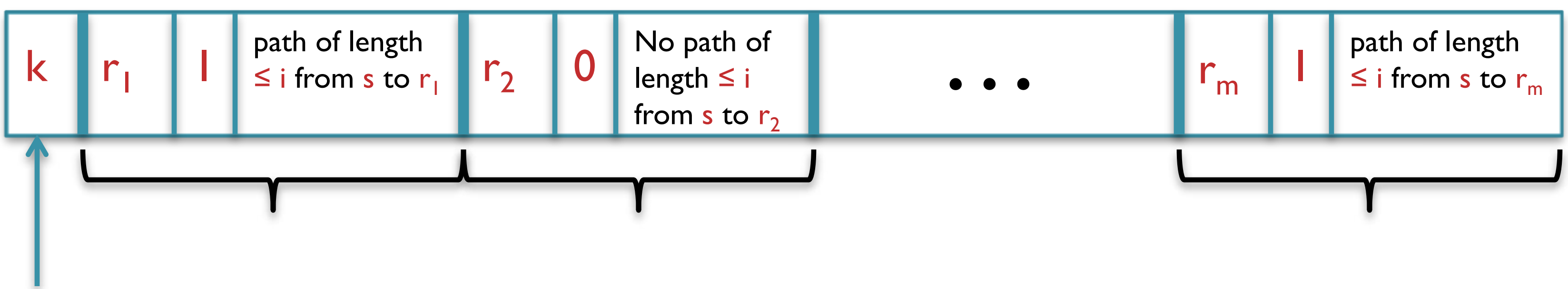
u_i looks like:



The claimed value of k_i.
$O(\log m)$ bits required.

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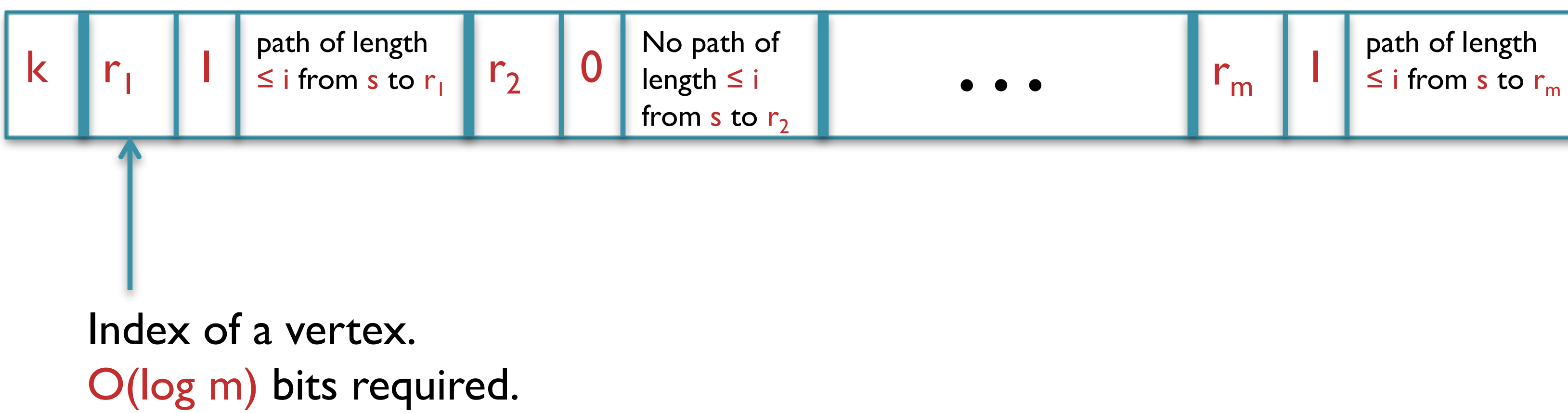

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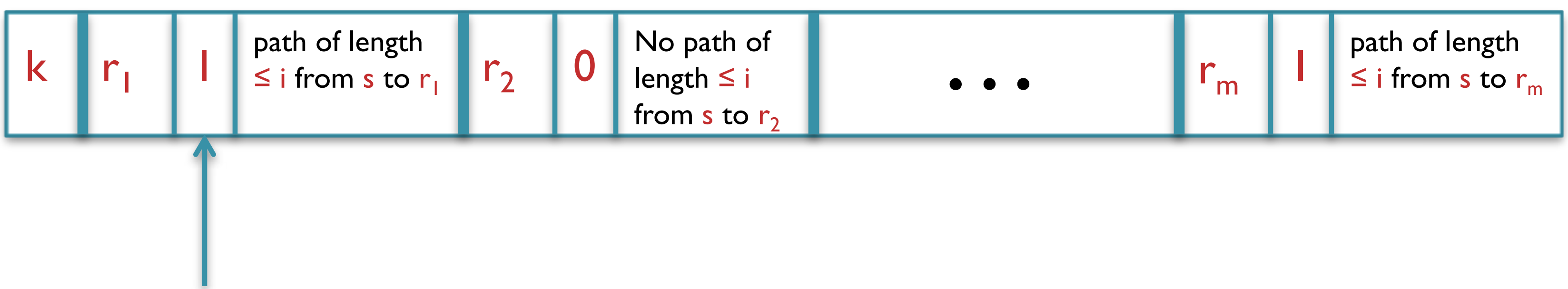

Indicator bit that indicates if r_1 is reachable from s by a path of length $\leq i$

NL = co-NL

- **Theorem.** (*Immerman-Szelepcsenyi 1987*) $\overline{\text{PATH}} \in$ NL.
- **Proof.** We'll design u_i assuming that $u_1, \ldots, u_{i-1}$ have already been constructed and M knows k_{i-1}. Let $r_1, \ldots r_m$ be the nodes of G s.t. $r_1 < r_2 < \ldots < r_m$. Then,

u_i looks like:

k	r_1	1	path of length $\leq i$ from s to r_1	r_2	0	No path of length $\leq i$ from s to r_2	. . .	r_m	1	path of length $\leq i$ from s to r_m

If indicator bit is 1 then give a path from s to r_1 of length $\leq i$. O(m log m) bits required for this.

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If indicator bit is 0 then give a certificate for absence of paths from s to r_2 of length $\leq i$. (how?)

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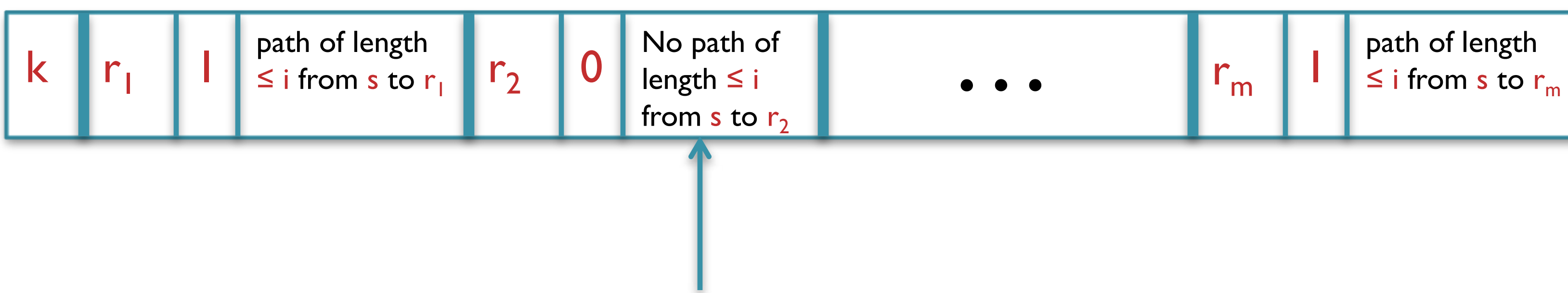

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If such certificates can be given using poly(m) bits then $|u_i| = \text{poly}(m)$

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- While reading u_i, M's work tape remembers the following info:

1. k_{i-1} and k,
2. the last read index of a vertex r_j

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- While reading u_i, M's work tape remembers the following info:

The moment M encounters a new vertex index r, it checks immediately if $r > r_j$. This ensures that M is not fooled by repeating info about the same vertex in u_i.

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2. the last read index of a vertex r_j

NL = co-NL

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While reading u_i, M keeps a count of the number of indicator bits that are 1 and finally checks if this number is k.

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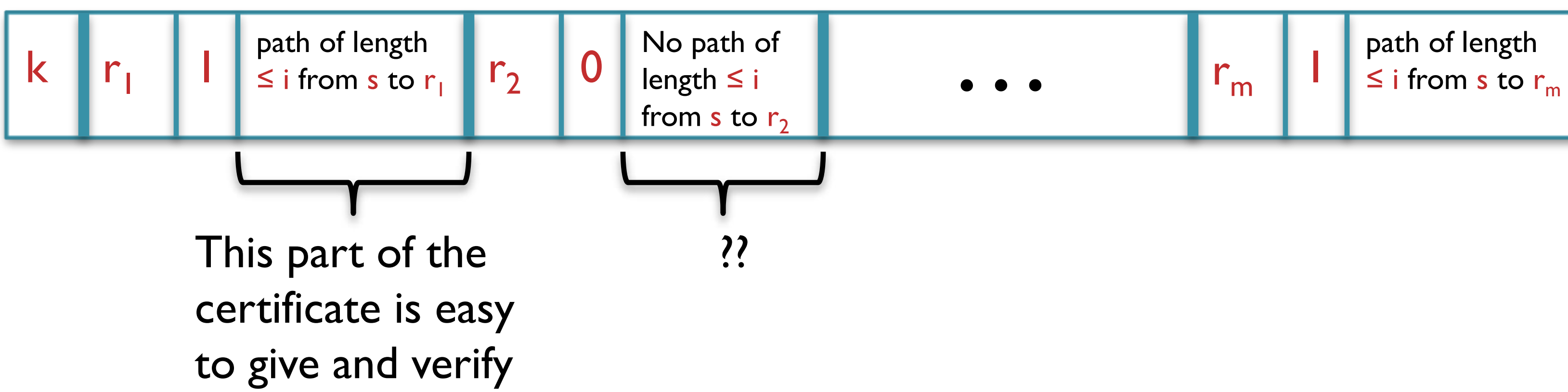
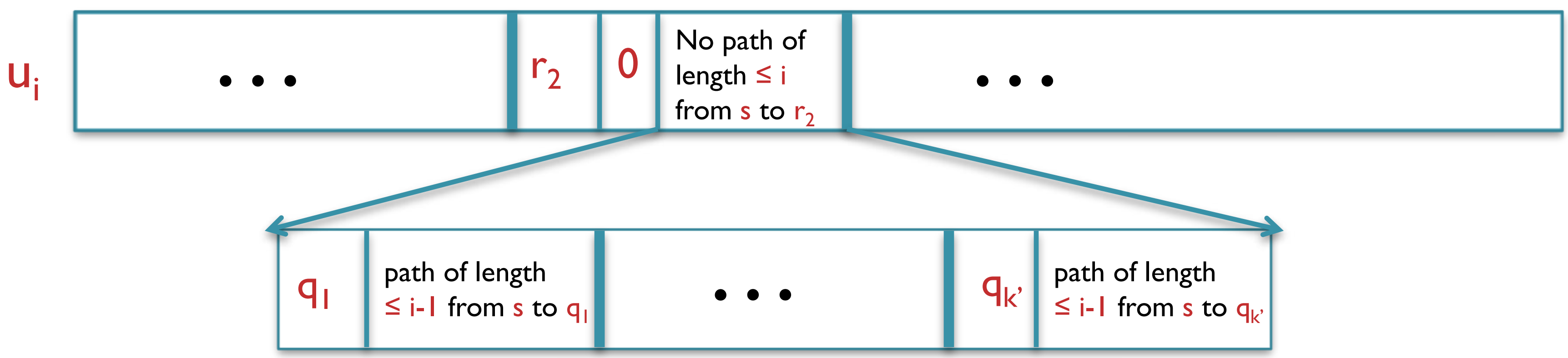


NL = co-NL

- Theorem. (*Immerman-Szelepcsenyi 1987*) $\overline{\text{PATH}} \in$ NL.
- Proof. Recall, M knows $k_{i-1} = k'$ (say) while reading u_i.



$$q_1 < q_2 < \ldots < q_{k'}$$

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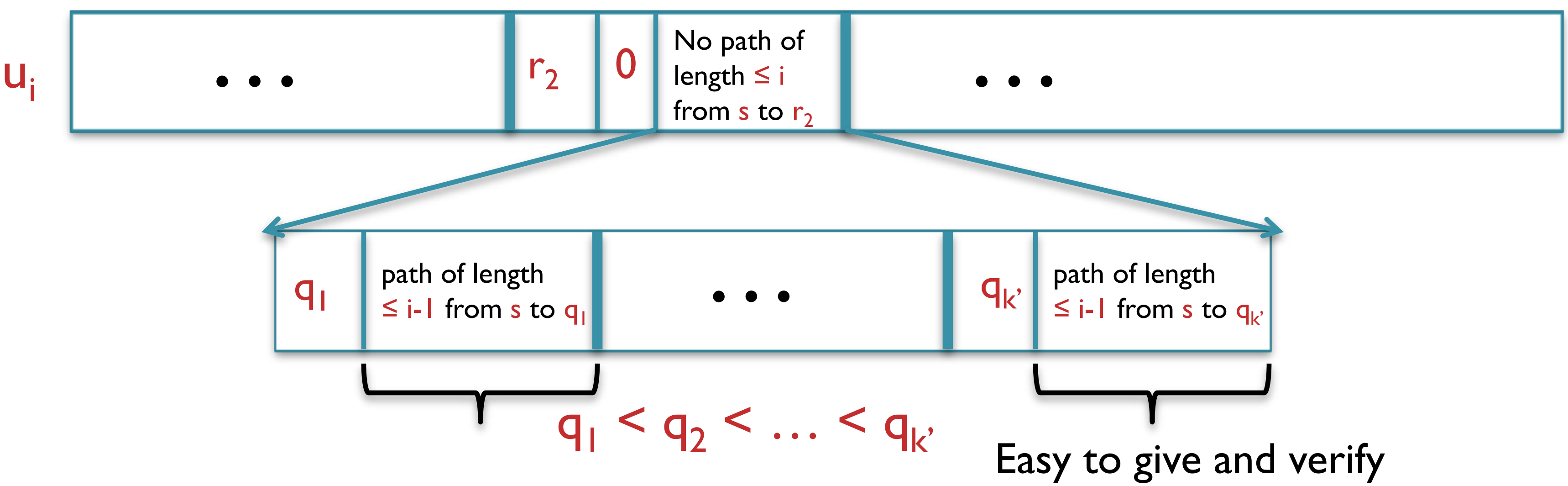

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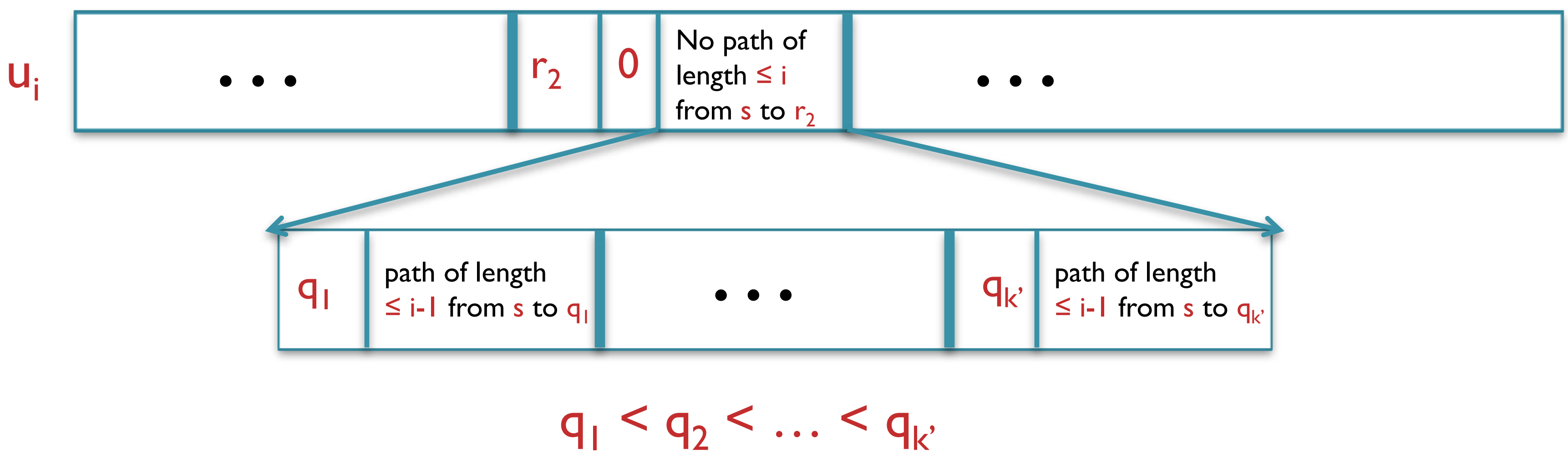

- While reading the 'No path…r_2' part of u_i, M remembers the last q_j read and checks that the next $q > q_j$. This ensures M is not fooled by repeating q's.

NL = co-NL

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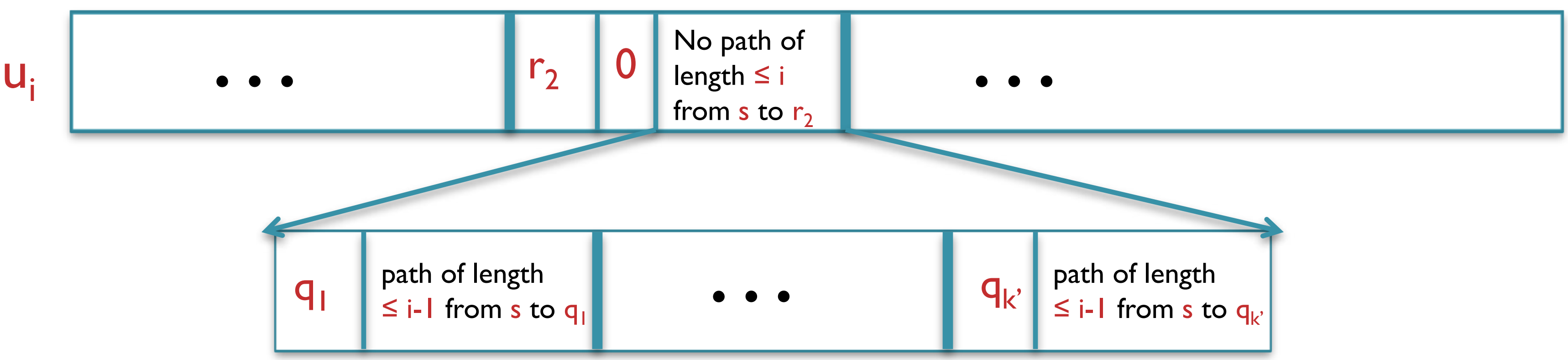
- **Proof.** Recall, M knows $k_{i-1} = k'$ (say) while reading u_i.



$q_1 < q_2 < \ldots < q_{k'}$

- For every $j \in [1, k_{i-1}]$, after verifying the path of length $\leq i-1$ from s to q_j, M checks that r_2 is <u>not</u> adjacent to q_j by looking at G's adjacency matrix.

NL = co-NL

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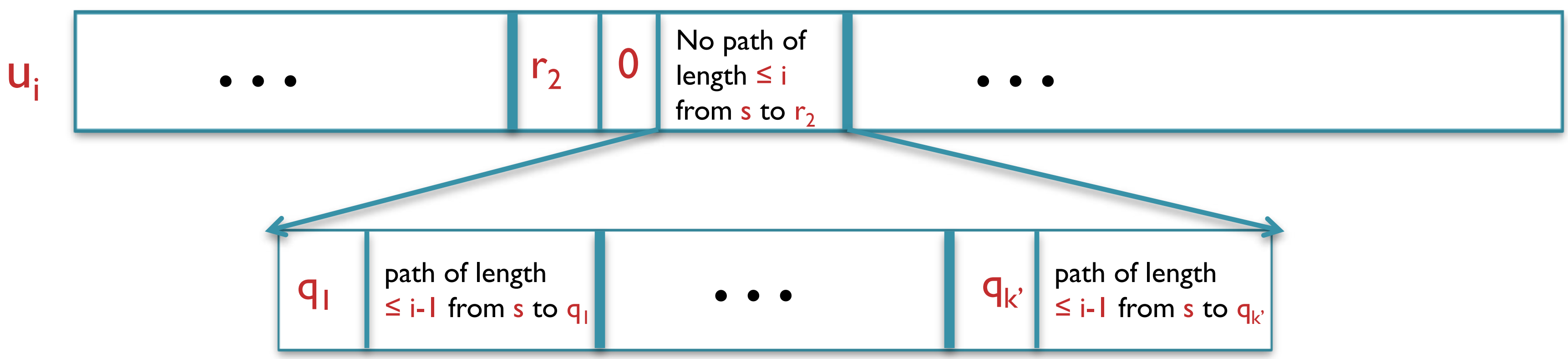

$q_1 < q_2 < \ldots < q_{k'}$

- At the end of reading the 'No path…r_2' part, M checks that the number of q's read is exactly k_{i-1}.

NL = co-NL

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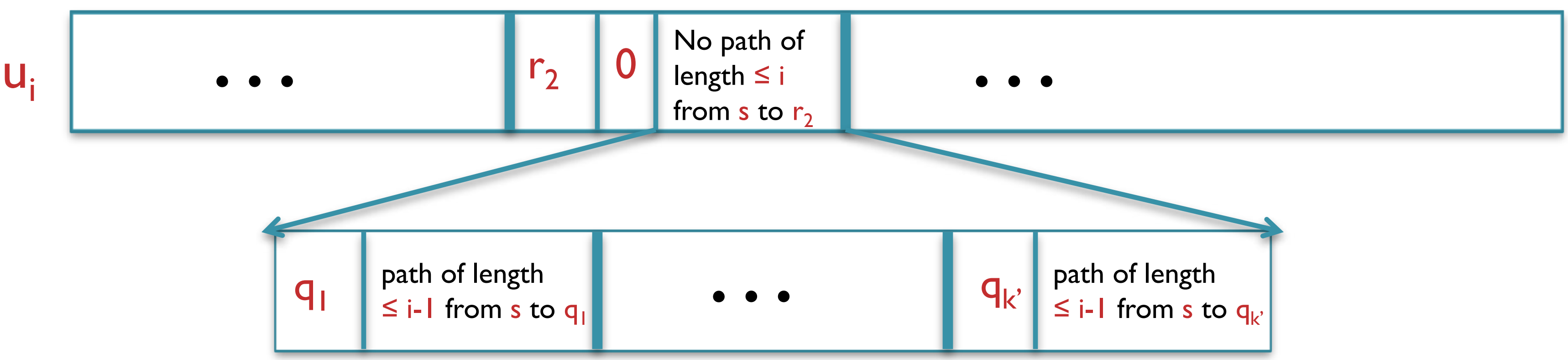

$q_1 < q_2 < \ldots < q_{k'}$

- This convinces M that there is no path of length $\leq i$ from s to r_2. Length of the 'No path…r_2' part of u_i is $O(m^2 \log m)$.

NL = co-NL

- Theorem. (*Immerman-Szelepcsenyi 1987*) $\overline{\text{PATH}} \in$ NL.
- Proof. So, after reading $(u_1, \ldots, u_m)$, the verifier M knows k_m, the number of vertices reachable from s.

- The v part of the certificate u is similar to the 'No path…r_2' part of u_i described before. The details here are easy to fill in (*homework*).

- We stress again that M is able to verify nonexistence of a path between s and t by <u>reading u once</u> from left to right and never moving its head backward.

NL = co-NL

- Hence, both PATH and

  $\text{PATH} \in \text{NL} \subseteq \text{SPACE}((\log n)^2)$

  by Savitch's theorem.