Computational Complexity Theory

Lecture 25: Complexity of Counting

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Natural counting problems

- What is the complexity of the following problems?

- #SAT: Count the number of satisfying assignments of a given Boolean circuit/CNF.

- #HAMCYCLE: Count the number of Hamiltonian cycles in an undirected graph.

- Observation. The above problems are NP-hard.

Natural counting problems

- What is the complexity of the following problems?

- #PerfectMatching: Count the number of perfect matchings in a bipartite graph.

- #CYCLE: Count the number of simple cycles in a directed graph.

- Observation. The corresponding decision problems are in P.

Natural counting problems

- What is the complexity of the following problems?

- #PATH: Count the number of simple paths between two vertices in a connected graph.

- #SPANTREE: Count the number of spanning trees in a connected graph.

- Observation. The corresponding decision problems are trivial.

An easy counting problem

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- Proof sketch. Let G be an n-vertex connected graph without self loops. Label the vertices by {1,…, n}.

- Definition. The *Laplacian matrix* of G is an n x n matrix L_G defined as

$$L_G(i,j) = \deg(i) \quad \text{if } i = j,$$
$$= -1 \quad \text{if there's an edge } (i,j) \text{ in } G,$$
$$= 0 \quad \text{otherwise.}$$

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- Definition. The *Laplacian matrix* of G is an $n \times n$ matrix L_G defined as $L_G = D_G - A_G$, where D_G is the degree matrix and A_G the adjacency matrix of G.

- Observation. It is easy to compute L_G from A_G.

An easy counting problem

- Theorem. (*Kirchhoff 1847*) #SPANTREE is in FP.

- Proof sketch. Let G be an n-vertex connected graph without self loops. Label the vertices by {1,…, n}.

- *Kirchhoff's matrix-tree theorem* states that

  no. of spanning trees of G = any cofactor of L_G.

- (i,j) *cofactor* of L = $(-1)^{i+j}$. det(submatrix of L obtained by deleting the i-th row and the j-th column from L).

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- Corollary. As determinant computation is in (functional) NC, #SPANTREES is in (functional) NC.

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- Theorem. #CYCLE is in NP-hard.

- Lesson. A counting problem can be hard even if the corresponding decision problem is in P.

A hard counting problem

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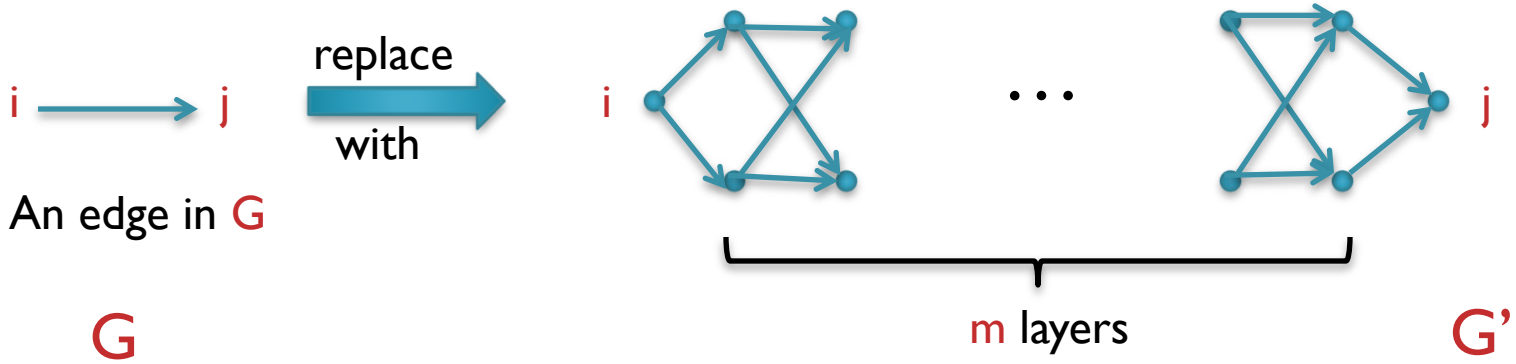
- Proof. We will give a poly-time reduction from the Hamiltonian cycle problem to the #CYCLE problem.

A hard counting problem

- Theorem. #CYCLE is in NP-hard.

- Proof. Let G be an n-vertex digraph. We'll efficiently construct a new graph G' from G s.t. the presence of a Hamiltonian cycle in G can be readily derived from the number of cycles in G'. Construction of G' :



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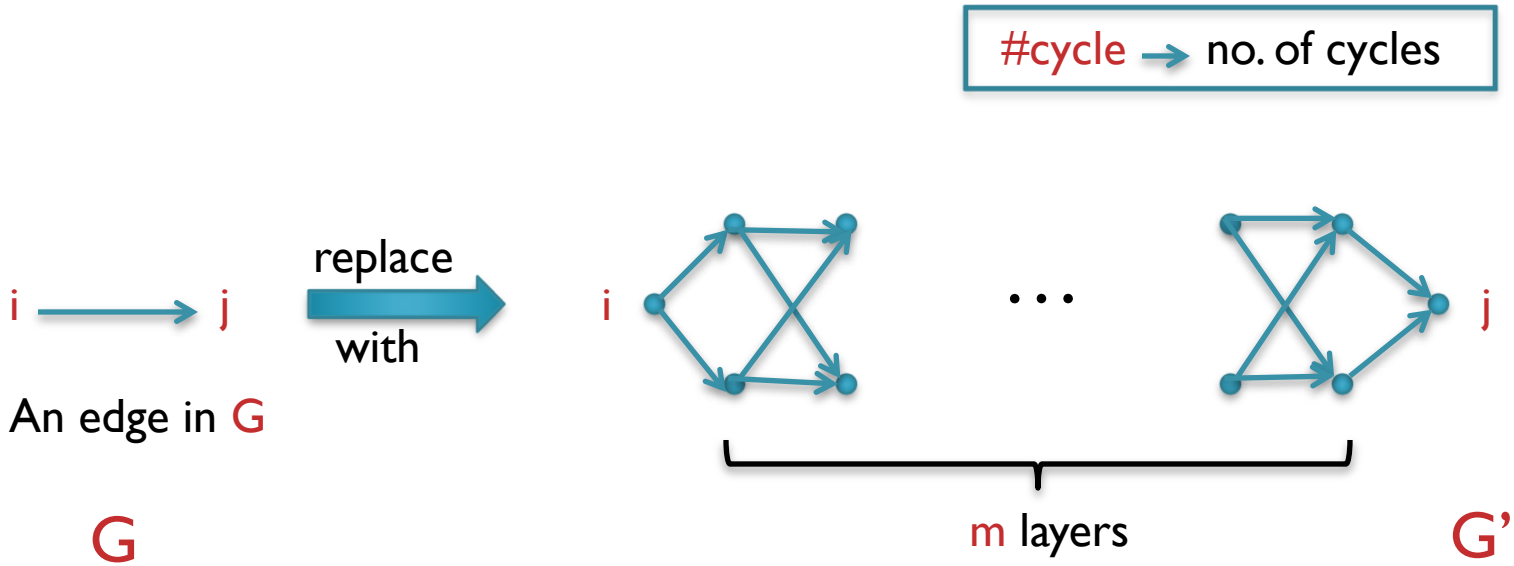

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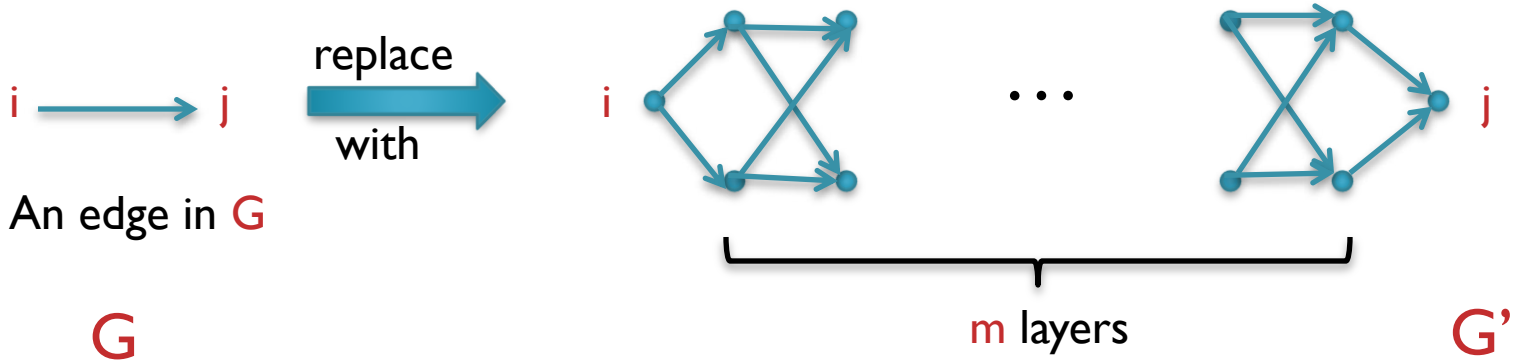
  #cycle(G') ≤ $n^{n-1}.2^{m(n-1)}$.



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- If we choose m such that $n^{n-1}.2^{m(n-1)} < 2^{mn}$, then we can find out if G has a HC from #cycle(G').
- Set $m = n^2$.

Class #P

- Definition. We say a function $f: \{0,1\}^* \to \mathbb{N}$ is in #P if there's a poly-time TM M and a polynomial function $p: \mathbb{N} \to \mathbb{N}$ such that for every $x \in \{0,1\}^*$,

$$f(x) = \left|\{u \in \{0,1\}^{p(|x|)} : M(x,u) = 1\}\right| .$$

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- Observation. Problems #SAT, #HAMCYCLE, #PerfectMatching, #CYCLE, #PATH and #SPANTREE are in #P.
- In fact, with every language in NP we can associate a counting problem that is in #P.

#P-completeness

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is #P = FP ?

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- In other words, for every $x \in \{0,1\}^*$, we can compute $g(x)$ in polynomial time using oracle access to f.

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- Observation. If a #P-complete language is in FP then #P = FP.

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- Algorithm: On input x, convert M(x, ..) to a 3CNF ϕ_x using Cook-Levin theorem. Give ϕ_x as input to the #SAT oracle. Output whatever the oracle outputs.

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Note: Only <u>one query</u> to the oracle. Resembles a poly-time Karp reduction.

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$$g(x) = \left|\{u \in \{0,1\}^{p(|x|)} : M(x, u) = 1\}\right| .$$

- Correctness: Follows from the fact that the Cook-Levin reduction is *parsimonious*, i.e.,

$$\left|\{u \in \{0,1\}^{p(|x|)} : M(x, u) = 1\}\right| = \#\phi_x .$$

The no. of satisfying assignments of ϕ_x.

Natural #P-complete problems

- Theorem. #HAMCYCLE is #P-complete.

- Most (all?) NP-complete problems known till date have defining verifiers such that the corresponding counting problems are #P-complete.

- Open. Does every NP-complete problem have a defining verifier such that the corresponding counting problem is #P-complete ?

Issue: The reduction that shows NP-completeness of a problem needn't have to be parsimonious.

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- Theorem. *(Valiant 1979)* #PerfectMatching is #P-complete.
- Proof. We'll see a proof later.

Relation between #P and other classes

- Observation. #P $\subseteq$ PSPACE.

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- Observation. #P $\subseteq$ PSPACE.
- Also, PH $\subseteq$ PSPACE. How does #P relate to PH ?
- Theorem. *(Toda 1991)* PH $\subseteq P^{\#SAT}$.
- Hence, #P is *harder* than PH.

Approximations of #P functions

- Observation. If #P = FP, then P = NP.
- Open. Does P = NP imply #P = FP ?

- But, we do know that P = NP implies every #P problem has a randomized polynomial-time *approximation* algorithm.

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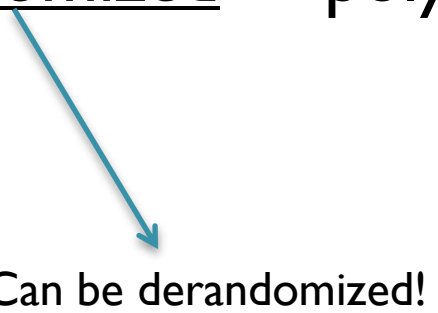

Approximations of #P functions

- Definition. A function $f: \{0,1\}^* \longrightarrow \mathbb{N}$ has a *Fully Polynomial-time Randomized Approximation Scheme* (FPRAS) if for every $\varepsilon, \delta > 0$, there's a PTM M such that for every $x \in \{0,1\}^*$,
  - $(1-\varepsilon).f(x) \leq M(x) \leq (1+\varepsilon).f(x)$ with prob. $\geq 1-\delta$,
  - M runs in $\text{poly}(|x|, \varepsilon^{-1}, \log \delta^{-1})$ time.

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- Remark. In fact the above FPRAS can be replaced by a FPTAS (*Fully Poly-Time Approximation Scheme*).

Approximations of #P functions

- Some #P-complete problems do admit FPRAS unconditionally!

- Theorem. *(Jerrum, Sinclair, Vigoda 2001)* #PerfectMatching has a FPRAS.

- Remark. No derandomization of this algorithm is known!

Approximations of #P functions

- Some #P-complete problems do admit FPRAS unconditionally!

- Theorem. *(Jerrum, Sinclair, Vigoda 2001)* Permanent of a square matrix with non-negative entries has a FPRAS.

- If $X = (x_{ij})_{i,j \in n}$ then $\mathrm{Perm}(X) = \sum_{\sigma \in S_n} \prod_{i \in [n]} x_{i\,\sigma(i)}$.

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- Note. If B_G is the biadjacency matrix of a bipartite graph G, then $\mathrm{Perm}(B_G) = \#\mathrm{PerfectMatchings}(G)$.

0/1 matrix