

Lec 28: Class PP and $\oplus P$; Valiant-Vazirani theorem

A couple of observations.

- Obs 1: If there's a NTM M that decides L in time $T(n)$, then there's a NTM M' that decides L in time $T'(n) = O(T(n))$ and every computation path of M' on i/p x has length exactly $T'(|x|)$.
- Obs 2: If there's a PTM M that has run-time $T(n)$, then there's a PTM M' that has run-time $T'(n) = O(T(n))$ s.t. $\Pr[M'(x)=1] = \Pr[M(x)=1] \quad \forall x \in \{0,1\}^*$, and every computation path of M' on i/p x has length exactly $T'(|x|)$.

Class PP (Probabilistic Polynomial Time)

- It is the decision version of the class $\#P$.
- Defn. 1: A language L is in PP if there's a poly-time PTM M s.t. $x \in L \iff \Pr[M(x)=1] > \frac{1}{2}$.

Equivalent Definitions

- Defn 2: L is in PP if there's a poly-time NTM M s.t. $x \in L \iff$ majority (i.e., strictly $> \frac{1}{2}$) of the computation paths of M on i/p x are accepting.
- Defn 3: L is in PP if there's a poly-time DTM M and a polynomial function $p(\cdot)$ s.t.
$$x \in L \iff \left| \{v \in \{0,1\}^{p(|x|)} : M(x,v)=1\} \right| > \frac{1}{2} \cdot 2^{p(|x|)}.$$

PP-completeness

- Defn.: A language $L \in PP$ is PP-complete if $L' \leq_p L$ for every $L' \in PP$.
 - Let $\text{MajSAT} := \{ \text{Boolean ckt. } \varphi(x_1, \dots, x_n) : \# \varphi > 2^{n-1} \}$.
 - Obs.: MajSAT is PP-complete under poly-time (Karp) reduction.
 - Lemma: $P^{\text{MajSAT}} = P^{\#SAT}$ (i.e. $P^{PP} = P^{\#P}$).
 - Proof: $\Rightarrow \text{MajSAT} \in P^{\#SAT}$ (easy).
 $\Leftarrow \#SAT \in P^{\text{MajSAT}}$
- Notation. For an $x \in \{0,1\}^*$, $\text{Int}(x)$ will denote the integer corresponding to x .

- Proof: (contd.) Let $\varphi(x_1, \dots, x_n)$ be the i/p. ckt. We wish to compute $\# \varphi$. Let $M(b_1, \dots, b_n, x_1, \dots, x_n)$ be a DTM that outputs 1 iff $\text{Int}(\underline{x}) < \text{Int}(\underline{b})$.

$$M(\underline{b}, \underline{x}) \xrightarrow[\text{by fixing } \underline{b}]{\text{Apply Cook-Levin}} \psi_{\underline{b}}(\underline{x}) \text{ (a Bool. ckt.)}$$

- Obs: $\# \psi_{\underline{b}}(\underline{x}) = \text{Int}(\underline{b}) \in [0, 2^{n-1}]$.

- Define, $\Gamma_{\underline{b}}(\underline{z}, \underline{x}) := (\underline{z} \wedge \varphi(\underline{x})) \vee (\neg \underline{z} \wedge \psi_{\underline{b}}(\underline{x}))$.

$$\Rightarrow \boxed{\# \Gamma_{\underline{b}}(\underline{z}, \underline{x}) = \# \varphi + \text{Int}(\underline{b})}$$

- Proof (contd.). Query $\Gamma_{\underline{b}}(\underline{z}, \underline{x})$ to the MajSAT oracle to check if $\# \Gamma_{\underline{b}}(\underline{z}, \underline{x}) > \frac{1}{2} \cdot 2^{n+1} = 2^n$.
- Now use binary search on $\{0, 1\}^n$ to find the smallest $\underline{b}$ s.t. $\# \varphi + \text{Int}(\underline{b}) = 2^n \Rightarrow \# \varphi = 2^n - \text{Int}(\underline{b})$. ▣

Today's theorem: $PH \subseteq P^{\#SAT} = P^{\text{MajSAT}}$.

Clearly, $NP, \text{co-NP} \subseteq P^{\#SAT}$.

Class $\oplus P$ (Parity P)

- Defn.: A language L is in $\oplus P$ if there's a NTM M s.t. $x \in L \Leftrightarrow$ the number of accepting paths of M on i/p x is odd.
 - $\oplus SAT := \{ \text{Boolean ckt. } \varphi : \# \varphi \text{ is odd} \}$.
 - Obs.: $\oplus SAT$ is $\oplus P$ -complete under poly-time (Karp) reduction.
 - Obs.: $\text{co-}\oplus P = \oplus P$, i.e., $\oplus P$ is closed under complementation.
- Proof: $\overline{\oplus SAT} := \{ \text{Boolean ckt. } \varphi : \# \varphi \text{ is even} \}$ is co- $\oplus P$ -complete.

- We wish to show that $\overline{\oplus SAT} \in \oplus P$.

- Let φ be the i/p formula. Define,

$$\psi(z, \underline{x}) := (z \wedge \varphi(\underline{x})) \vee (\neg z \wedge x_1 \wedge \dots \wedge x_n).$$

$$\Rightarrow \# \psi(z, \underline{x}) = \# \varphi(\underline{x}) + 1. \quad (\text{Denote } \psi(z, \underline{x}) \text{ as } \underline{\varphi+1})$$

- Therefore, $\varphi(\underline{x}) \in \overline{\oplus SAT} \iff \psi(z, \underline{x}) \in \oplus SAT$.

- Think of a NTM that on i/p φ , at first forms ψ and then guesses z and $\underline{x}$ and outputs $\psi(z, \underline{x})$.



The $\oplus$ quantifier

- We will treat $\oplus$ as a quantifier, just like $\exists$ & $\forall$.
- **Defn:** For a Boolean ckt. $\varphi(x_1, \dots, x_n)$, we say

$\bigoplus_{\underline{x}} \varphi(\underline{x})$ is true if $\# \varphi$ is odd.

- We can define the problem $\oplus$ SAT, equivalently, as the set of all true quantified Boolean ckt. of the form $\bigoplus_{\underline{x}} \varphi(\underline{x})$.

Today's theorem: Proof outline

- Step 1: Give a randomized poly-time reduction from PH to $\oplus$ SAT.
- Step 2: (Derandomization of Step 1). Give a deterministic poly-time reduction from PH to $\#$ SAT.
- Open: $NP \subseteq P^{\oplus SAT} ?$
- Proof of Step 1 uses the Valiant-Vazirani theorem.

Valiant - Vazirani theorem

- $USAT := \{ \text{Boolean ckt } \varphi : \# \varphi = 1 \}$
 $\uparrow$
unique
- Obs: $USAT \subseteq \oplus SAT$.
- Theorem (VV'86): There is a randomized poly-time reduction f s.t. for every n -variate Boolean ckt. φ , the following holds:
$$\varphi \in SAT \Rightarrow \Pr [f(\varphi) \in USAT] \geq \frac{1}{8n},$$
$$\varphi \notin SAT \Rightarrow \Pr [f(\varphi) \notin SAT] = 1.$$

- Corollary 1: There is a randomized poly-time reduction from SAT to $\oplus$ SAT with success probability $\geq \frac{1}{8n}$.

↑ This success prob. can be boosted.
We'll see how later.

- VV Lemma: Let $f_{n,k}$ be a family of pairwise independent hash fns. from $\{0,1\}^n$ to $\{0,1\}^k$, and $S \subseteq \{0,1\}^n$ be such that $2^{k-2} \leq |S| \leq 2^{k-1}$. Then,
$$\Pr_{h \in f_{n,k}} \left[\text{there's a unique } \underline{x} \in S \text{ s.t. } h(\underline{x}) = 0^k \right] \geq \frac{1}{8}.$$

Proof of the Valiant-Vazirani Lemma

- For every $\underline{x}, \underline{x}' \in S$, $\underline{x} \neq \underline{x}'$,
 - $\triangleright \Pr_h[h(\underline{x}) = 0^k] = \Pr_h[h(\underline{x}') = 0^k] = \frac{1}{2^k}$.
 - $\triangleright \Pr_h[h(\underline{x}) = 0^k \wedge h(\underline{x}') = 0^k] = \frac{1}{2^{2k}}$.
- Let $N := \text{No. of } \underline{x} \in S \text{ s.t. } h(\underline{x}) = 0^k$. We would like to lower bound $\Pr_h[N = 1]$.
- For the rest of the proof, we'll denote $\Pr_h$ as $\Pr$.

• Proof (contd.). Observe that

$$\Pr[N=1] + \Pr[N \geq 2] = \Pr[N \geq 1]$$

$$\Rightarrow \Pr[N=1] = \Pr[N \geq 1] - \Pr[N \geq 2]$$

Need to lower bound $\uparrow$

$\uparrow$ Need to upper bound

$$\bullet \Pr[N \geq 2] \leq \binom{|S|}{2} \cdot \frac{1}{2^{2k}} \quad (\text{by union bound}).$$

$$\bullet \Pr[N \geq 1] \geq |S| \cdot \frac{1}{2^k} - \binom{|S|}{2} \cdot \frac{1}{2^{2k}}$$

(by inclusion - exclusion principle)

• Therefore,

$$\Pr[N=1] \geq |S| \cdot \frac{1}{2^k} - 2 \cdot \binom{|S|}{2} \cdot \frac{1}{2^{2k}}$$

$$\geq |S| \cdot \frac{1}{2^k} - |S|^2 \cdot \frac{1}{2^{2k}}$$

$$\geq \frac{1}{8}.$$

• As $2^{k-2} \leq |S| \leq 2^{k-1}$,

$$\frac{1}{4} \leq \frac{|S|}{2^k} \leq \frac{1}{2}.$$



Proof of the Valiant-Vazirani theorem

- let M be a DTM that takes $\text{inp}: k \in \mathbb{N}, h \in \mathcal{H}_{n,k}, x \in \{0,1\}^n$ and outputs 1 iff $h(x) = 0^k$.

$$M(k, h, x) \xrightarrow[\text{by fixing } k, h]{\text{Cook-Levin reduction}} \psi_{k,h}(x) \text{ (Bool. cht.)}$$

- Consider the following randomized reduction f .

$$\varphi(x) \xrightarrow[\substack{1. \text{ Pick } k \in_r \{2, \dots, n+1\} \\ 2. \text{ Pick } h \in_r \mathcal{H}_{n,k}}]{f} \varphi(x) \wedge \psi_{k,h}(x) = f(\varphi)$$

Proof of the Valiant-Vazirani theorem (contd.)

- Obs: If $\varphi \notin \text{SAT}$ then $f(\varphi) \notin \text{SAT}$.
- Obs: If $\varphi \in \text{SAT}$ then $f(\varphi) \in \text{USAT}$ w.p. $\geq \frac{1}{8n}$.

Proof: Let S be the set of satisfying assignments of φ . Observe, $2^0 \leq |S| \leq 2^n$.

- With prob. $\frac{1}{n}$, the reduction function f chooses the "right" k , i.e., the chosen k satisfies $2^{k-2} \leq |S| \leq 2^{k-1}$.

- Therefore, conditional on the "right" choice of k ,
$$\Pr_h \left[\text{there's a unique } \underline{x} \in S \text{ s.t. } h(\underline{x}) = 0^k \right] \geq \frac{1}{8}$$

(by the Valiant-Vazirani lemma)

- $\Pr_{k,h} \left[f(\varphi) \in \text{USAT} \right] \geq \frac{1}{8n} .$

