Computational Complexity Theory

Lecture 8: Time Hierarchy Theorem; Ladner's theorem

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Recap: Diagonalization

- *Diagonalization* refers to a class of techniques used in complexity theory to separate complexity classes.
- These techniques are characterized by <u>two</u> main features:

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- These techniques are characterized by <u>two</u> main features:

    1. There's a universal TM U that when given strings α and x, simulates M_α on x with only a <u>*small*</u> overhead.

        If M_α takes T time on x then U takes $O(T \log T)$ time to simulate M_α on x.

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- These techniques are characterized by <u>two</u> main features:

  1. There's a universal TM U that when given strings α and x, simulates M_α on x with only a *small* overhead.

  2. Every string represents some TM, and every TM can be represented by *infinitely many* strings.

Time Hierarchy Theorem

- An application of Diagonalization

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be <u>time-constructible</u> functions s.t.,

  $$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. *(Hartmanis & Stearns 1965)*

  $$\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$$

- Theorem. $\mathrm{P} \subsetneq \mathrm{EXP}$

- This type of results are called <u>**lower bounds**</u>.

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  Task: Show that there's a language L decided by a TM D with time complexity $O(n^2)$ s.t., any TM M with runtime $O(n)$ cannot decide L.

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D's time steps not M_x's time steps.

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D outputs the **opposite** of what M_x outputs.

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  D runs in $O(n^2)$ time as n^2 is time-constructible.

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  Claim. There's no TM M with running time $O(n)$ that decides L (the language accepted by D).

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  - Suppose $M(x) = M_x(x) = b$.
  - D on input x, simulates M_x on x for $|x|^2$ steps. Since M_x stops within $c.|x|$ steps, D's simulation also stops within $c'.c.\,|x|.\log|x|$ steps.

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  - Hence, $D(x) = 1\text{-}b$.

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Contradiction! M does not decide L.

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- Theorem. $P \subsetneq EXP$

  Proof. Similar (homework)

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$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

- Theorem. $P \subsetneq EXP$
- **No** EXP-complete problem (under poly-time Karp reduction) is in P.

E.g., Decide if a TM halts in k steps; generalized versions of games such as chess, checkers, Go, etc.

Time Hierarchy Theorem

- Is there a natural problem that takes close to n^2 time?

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- Conjecture. ***No*** algorithm solves 3SUM in $O(n^{2-\varepsilon})$ time for some constant $\varepsilon > 0$.

- However, there's a $\sim O(n^2 / (\log n)^2)$ time algorithm for 3SUM. ("~" suppressing a poly(log log n) factor.)

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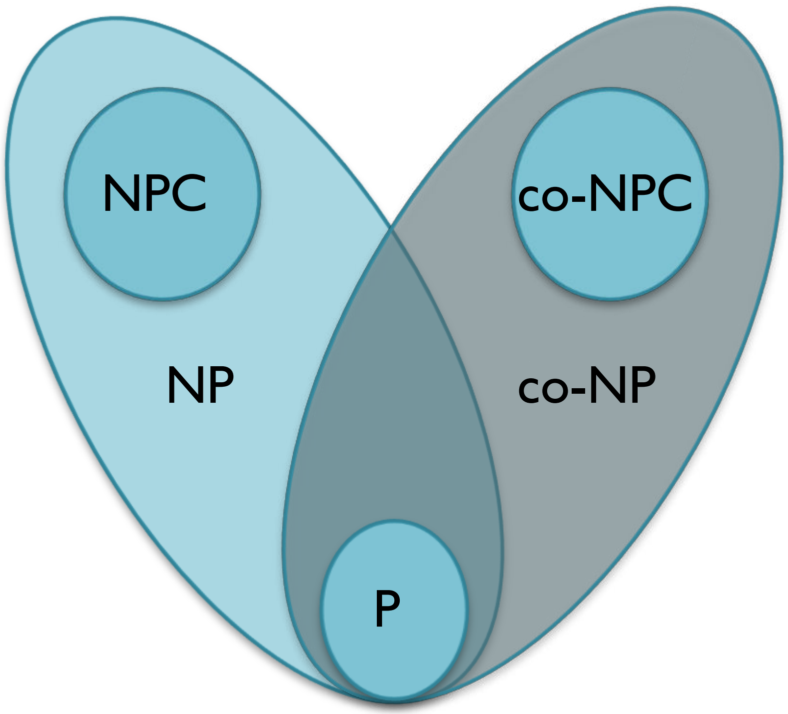
- kSUM: Given a list of n numbers, check if there exists k numbers in the list that sum to zero.
- Theorem *(Patrascu & Williams 2010)*. ETH implies kSUM requires $n^{\Omega(k)}$ time.

Revisiting NP∩co-NP



Conjecture: NP ≠ co-NP

⬇

P ≠ NP

General belief: P ≠ NP ∩ co-NP

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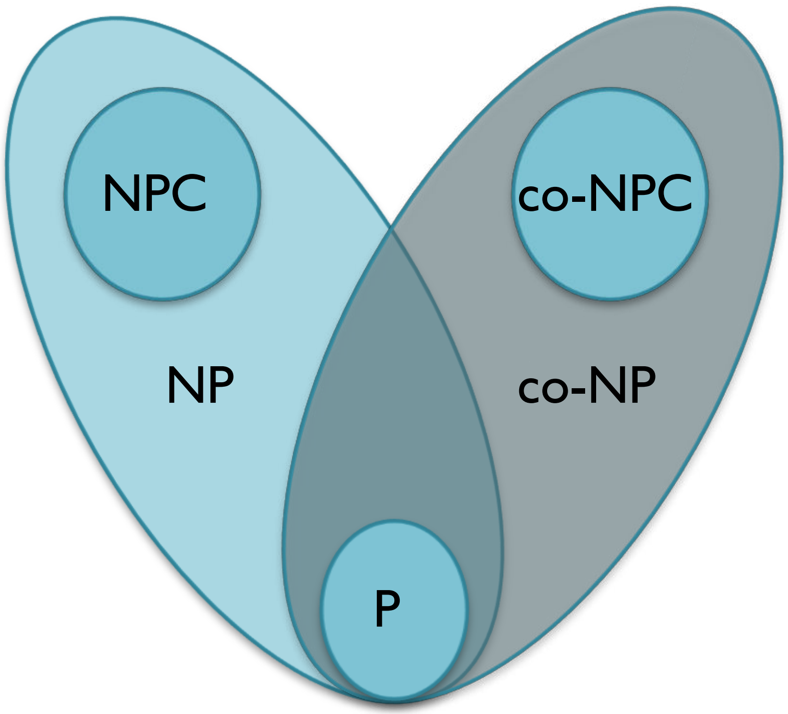

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Edmonds (1966)

…conjectured P = NP ∩ co-NP

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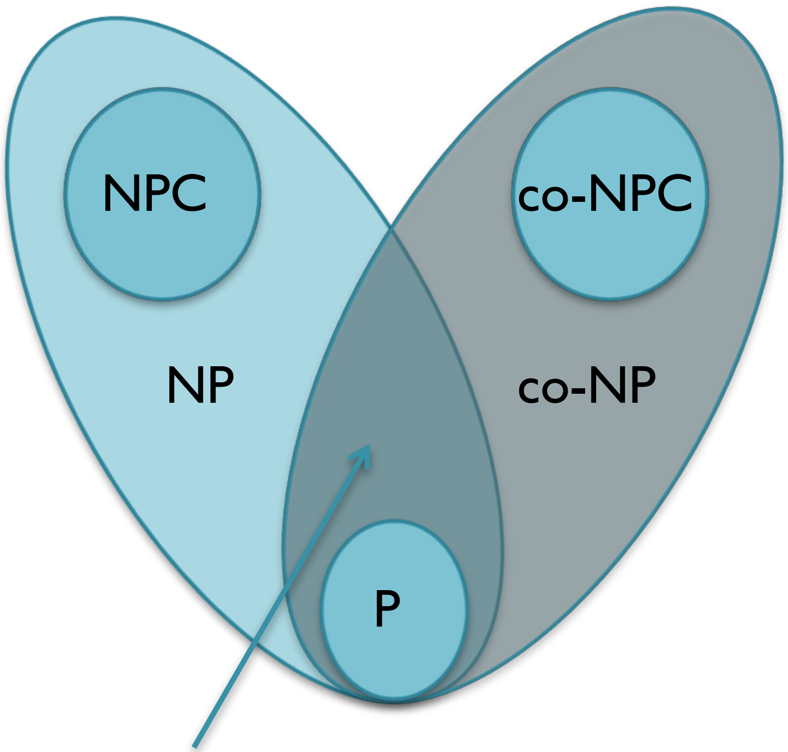

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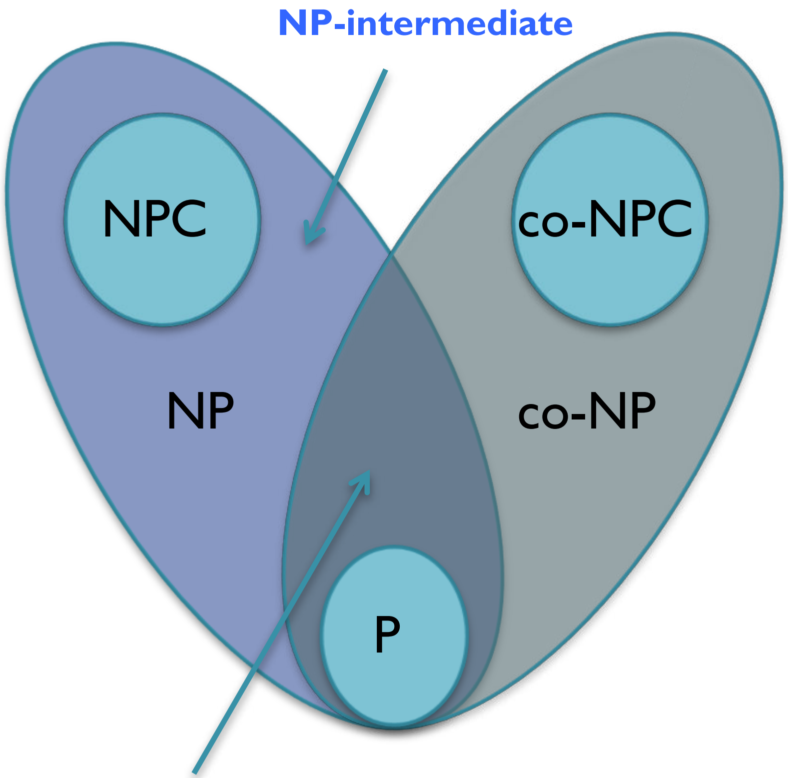
Check:
https://cstheory.stackexchange.com/questions/20021/reasons-to-believe-p-ne-np-cap-conp-or-not

- Integer factoring (FACT)
- Approximate shortest vector in a lattice

Ref: "*Lattice problems in NP∩co-NP*" by Aharonov & Regev (2005)

NP-intermediate problems



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Obs: If NP ≠ co-NP and FACT ∉ P then FACT is NP-intermediate.

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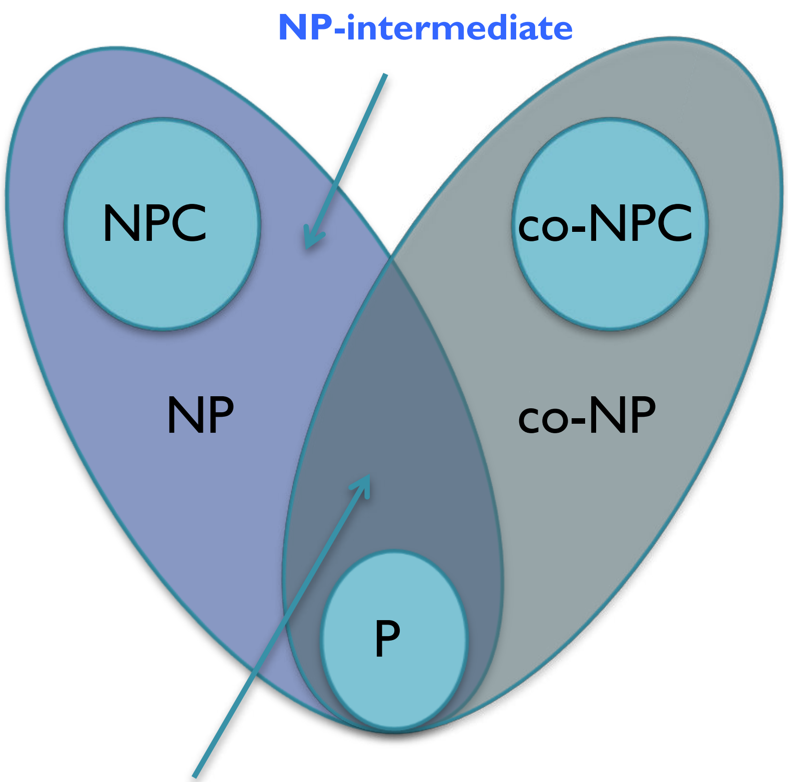

Conjecture: NP $\neq$ co-NP

$\Downarrow$

P $\neq$ NP

General belief: P $\neq$ NP $\cap$ co-NP

Obs: If NP $\neq$ co-NP and FACT $\notin$ P then FACT is NP-intermediate.

> Ladner's theorem: P $\neq$ NP implies existence of a NP-intermediate language.

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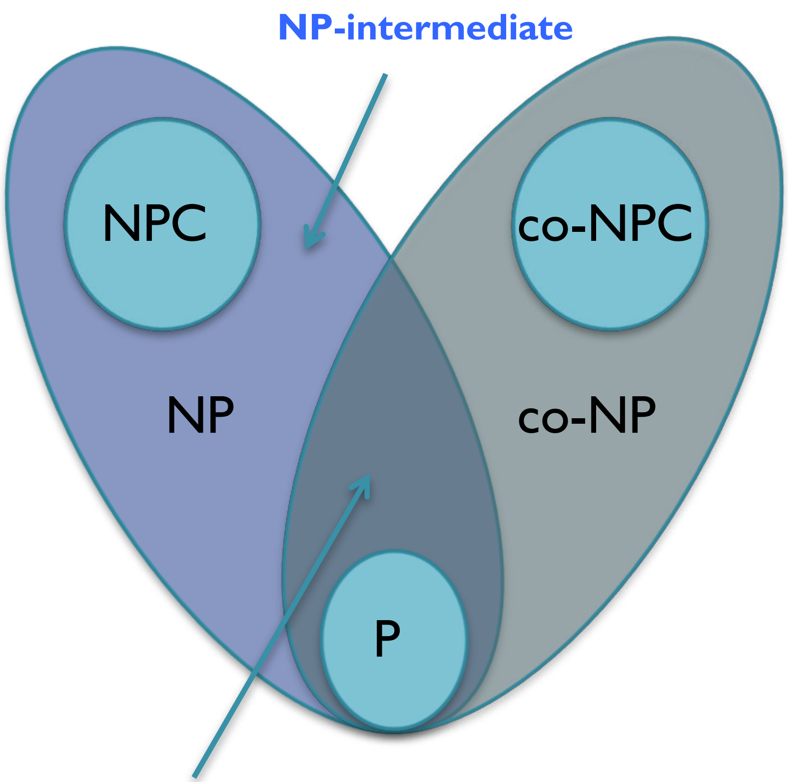

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Conjecture: NP ≠ co-NP

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P ≠ NP

General belief: P ≠ NP ∩ co-NP

Obs: If NP ≠ co-NP and FACT ∉ P then FACT is NP-intermediate.

Ladner's theorem: P ≠ NP implies existence of a NP-intermediate language.

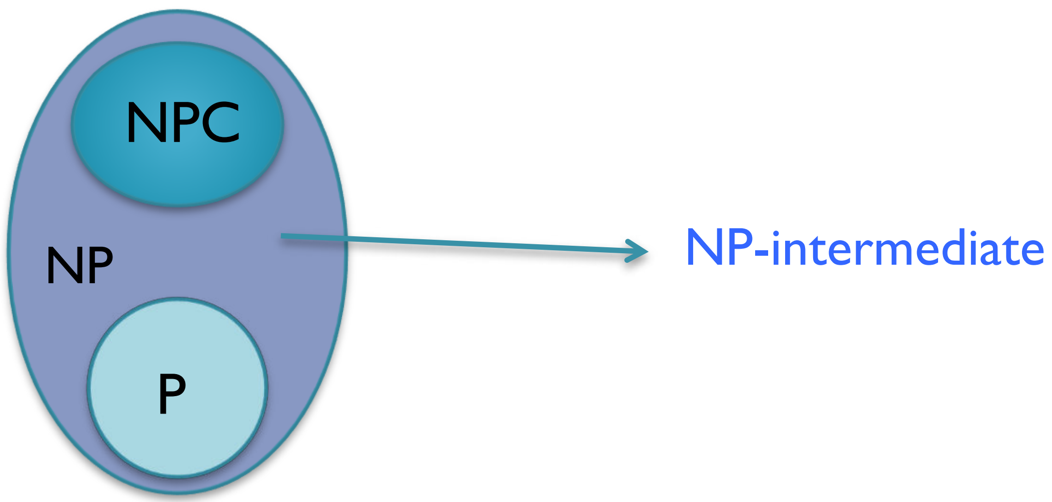
(proved using ***diagonalization***)

Ladner’s Theorem

- Another application of Diagonalization

NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.



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- Theorem. *(Ladner 1975)* If $P \neq NP$ then there is a NP-intermediate language.

NP-intermediate problems

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- Theorem. *(Ladner 1975)* If P ≠ NP then there is a NP-intermediate language.

  Proof. A delicate argument using diagonalization.

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  Proof. Let H: $\mathbb{N} \rightarrow \mathbb{N}$ be a function.

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  Proof. Let $H: \mathbb{N} \rightarrow \mathbb{N}$ be a function.

  Let $SAT_H = \{\Psi 0 1^{m^{H(m)}} : \Psi \in SAT \text{ and } |\Psi| = m\}$

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  H would be defined in such a way that SAT_H is NP-intermediate (assuming $P \neq NP$)

Ladner's theorem: Constructing H

- Theorem. There's a function $H: N \rightarrow N$ such that

    1. $H(m)$ is computable from m in $O(m^3)$ time.

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     for every m

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  3. If $SAT_H \notin P$ then $H(m) \rightarrow \infty$ with m.

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Proof: Later (uses diagonalization).

Let's see the proof of Ladner's theorem assuming the existence of such a "special" H.

Ladner's theorem: Proof

$P \neq NP$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.

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length at most $m + 1 + m^C$

Ladner's theorem: Proof

$P \neq NP$

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- This implies a poly-time algorithm for SAT as follows:
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  - Compute $H(m)$, and construct the string $\phi\, 0\, 1^{m^{H(m)}}$
  - Check if $\phi\, 0\, 1^{m^{H(m)}}$ belongs to SAT_H.
- As $P \neq NP$, it must be that $SAT_H \notin P$.

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \to \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$SAT \leq_p SAT_H$ $\qquad\qquad$ $\phi \xmapsto{f} \Psi\, 0\, 1^k$

Ladner's theorem: Proof

$$P \neq NP$$

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$$\phi \overset{f}{\longmapsto} \Psi\, 0\, 1^k$$

$$|\phi| = n \qquad |\Psi\, 0\, 1^k| = n^c$$

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$$SAT \leq_p SAT_H$$

$$\phi \overset{f}{\longmapsto} \Psi\, 0\, 1^k$$

$|\phi| = n$ $\qquad$ $|\Psi\, 0\, 1^k| = n^c$

Let m_0 be the largest s.t. $H(m_0) \leq 2c$.

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
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Either $m \leq m_0$ (in which case the task reduces to checking if a constant-size Ψ is satisfiable),

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or $H(m) > 2c$ (as $H(m)$ tends to infinity with m).

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- On input ϕ, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.
- Hence, w.l.o.g. $|f(\phi)| \geq k > m^{2c}$

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- On input ϕ, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.
- Hence, $\sqrt{n} \geq m$.

Ladner's theorem: Proof

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Thus, checking if an n-size formula ϕ is satisfiable reduces to checking if a $\sqrt{n}$-size formula Ψ is satisfiable.

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Do this recursively! Only $O(\log \log n)$ recursive steps required.

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  - Compute $H(m)$ and check if $k = m^{H(m)}$.
  - Hence, $\sqrt{n} \geq m$. Also $\phi \in SAT$ iff $\Psi \in SAT$

- Hence SAT_H is not NP-complete, as $P \neq NP$.

Natural NP-intermediate problems ??

- Integer factoring
- Approximate shortest vector in a lattice
- Minimum Circuit Size Problem

  ("*Multi-output MCSP is NP-hard*", Ilango, Loff & Oliveira 2020)
- Graph isomorphism

  ("*GI in QuasiP time*", Babai 2015)