E0 224: Computational Complexity Theory
Indian Institute of Science
Assignment 2

Due date: Oct 20, 2024 **Total marks: 60**

1. (**3 marks**) Define polyL to be $\cup_{c>0}\mathsf{SPACE}(\log^c n)$. Steve's Class SC is defined to be the set of languages that can be decided by deterministic machines that run in polynomial time and $\log^c n$ space for some $c > 0$. It is an open problem whether PATH $\in \mathsf{SC}$. Why does Savitch's theorem not resolve this question? Is SC the same as $\mathsf{polyL} \cap \mathsf{P}$.

2. (**10 marks**) Prove that there exists a language B such that $\mathsf{NP}^B \neq \text{co-}\mathsf{NP}^B$.

3. (**7 marks**) Prove that in the read-once certificate definition of NL, if we allow the verifier machine to move its head back and forth on the certificate then the class being defined changes to NP.

4. (**6 marks**) If $\mathcal{S} = \{S_1, S_2, ..., S_m\}$ is a collection of subsets of a finite set U, the *VC dimension* of $\mathcal{S}$, denoted $VC(\mathcal{S})$, is the size of the largest set $X \subseteq U$ such that for every $X' \subseteq X$, there is an i for which $S_i \cap X = X'$. (We say that X is *shattered* by $\mathcal{S}$.)

   A Boolean circuit C succinctly represents collections $\mathcal{S}$ if S_i consists of exactly those elements $x \in U$ for which $C(i, x) = 1$. Finally,

   $$\text{VC-DIMENSION} = \{< C, k > \; : \; C \text{ represents a collection } \mathcal{S} \text{ such that } VC(\mathcal{S}) \geq k\}.$$

   Show that VC-DIMENSION $\in \Sigma_3$.

5. (**9 marks**) Prove that a language L is in NC^1 if and only if L is decided by a $q(n)$-size circuit family $\{C_n\}_{n\in\mathbb{N}}$, where q is a polynomial function and C_n is a Boolean *formula* for every $n \in \mathbb{N}$.

6. (**10 marks**) Linear programming (LP) is the problem of checking the feasibility of a system of linear inequality constraints over rationals. Prove that every language in P is logspace-reducible to LP. (In other words, LP is P-complete, and so, if LP is in NC, then $\mathsf{P} = \mathsf{NC}$.)

7. (**6+9 marks**) Prove that logspace uniform NC^1 is contained in L. Prove that $\mathsf{NL} \subseteq \mathsf{NC}$.