E0 224: Computational Complexity Theory
Indian Institute of Science
Assignment 3

Due date: Nov 14, 2024 **Total marks: 50**

1. (**4 marks**) Give a polynomial time algorithm that checks whether a given bipartite graph $G = (V, E)$ is contained in $\oplus$Perfect Matchings, where $\oplus$Perfect Matchings is the set of all bipartite graphs having odd number of perfect matchings.

2. (**5 marks**) Prove that for any $n \times n$ matrix $A = (a_{i,j})_{i,j \in [n]}$,
$$\text{perm}(A) = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i \in [n]} \left(\sum_{j \in S} a_{i,j} \right).$$
Use this to design an algorithm to compute the permanent in time $2^n \cdot \text{poly}(n)$.

3. (**4 marks**) Consider the following problem: Given an n-variate polynomial f in the form $\prod_{i \in [n]} \sum_{j \in [n]} a_{i,j} x_j$, where $a_{i,j}$ are integers, and $e_1, \ldots, e_n \in \mathbb{Z}_{\geq 0}$ s.t. $e_1 + \ldots + e_n = n$, compute
$$\frac{\partial^n f}{\partial x_1^{e_1} \partial x_2^{e_2} \cdots \partial x_n^{e_n}}.$$
Prove that the problem is $\#\mathsf{P}$-hard.

4. (**6 marks**) Prove that $\mathsf{ZPP} = \mathsf{RP} \cap \mathsf{co} - \mathsf{RP}$.

5. (**6 marks**) Let BPL be the logspace variant of BPP, i.e., a language L is in BPL if there is an $O(\log(n))$ space probabilistic Turing machine M such that $\Pr[M(x) = L(x)] \geq 2/3$. Prove that $\mathsf{BPL} \subseteq \mathsf{P}$.

6. (**7 marks**) Prove that $\mathsf{BP.NP}$ is in Σ_3.

7. (**9 marks**) Prove that $\overline{SAT} \in \mathsf{BP.NP}$ implies $\mathsf{PH} = \Sigma_3$.

8. (**9 marks**) Give a randomized algorithm that takes input two $n \times n$ matrices A and B with integer entries and does the following: If A and B are similar, then with high probability the algorithm outputs an $n \times n$ invertible matrix C with rational entries such that $CAC^{-1} = B$; otherwise it outputs 'A not similar to B'. Ensure that your algorithm runs in polynomial time.