Computational Complexity Theory

Lecture 14: Polynomial Hierarchy (contd.); Boolean Circuits; Karp-Lipton theorem

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Recap: Class Σ_i

- Definition. A language L is in Σ_i if there's a polynomial function q(.) and a poly-time TM M (the "verifier") s.t.

  $x \in L \iff \exists u_1 \in \{0,1\}^{q(|x|)}\ \forall u_2 \in \{0,1\}^{q(|x|)}\ Q_i u_i \in \{0,1\}^{q(|x|)}$

  s.t. $M(x,u_1,\ldots, u_i) = 1$,

  where Q_i is $\exists$ or $\forall$ if i is odd or even, respectively.

- Obs. $\Sigma_i \subseteq \Sigma_{i+1}$ for every i.

Recap: Polynomial Hierarchy

- **Definition.** A language L is in Σ_i if there's a polynomial function $q(.)$ and a poly-time TM M (the "verifier") s.t.

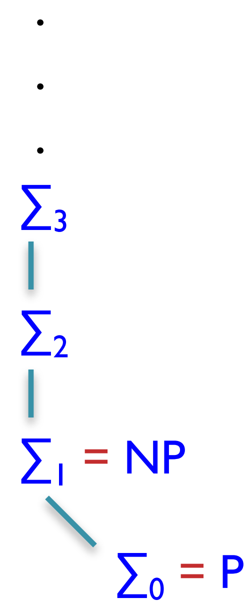
$$x \in L \Leftrightarrow \exists u_1 \in \{0,1\}^{q(|x|)} \; \forall u_2 \in \{0,1\}^{q(|x|)} \; Q_i u_i \in \{0,1\}^{q(|x|)}$$

$$\text{s.t. } M(x,u_1,\ldots,u_i) = 1,$$

where Q_i is $\exists$ or $\forall$ if i is odd or even, respectively.

- **Definition.** *(Meyer & Stockmeyer 1972)*

$$PH = \bigcup_{i \in \mathbf{N}} \Sigma_i .$$



Recap: Class Π_i

- Definition. Π_i = co-Σ_i = $\{ L : \bar{L} \in \Sigma_i \}$.

- Obs. A language L is in Π_i if there's a polynomial function q(.) and a poly-time TM M (the "verifier") s.t.

  $$x \in L \iff \forall u_1 \in \{0,1\}^{q(|x|)}\ \exists u_2 \in \{0,1\}^{q(|x|)}\ \ Q_i u_i \in \{0,1\}^{q(|x|)}$$

  $$\text{s.t. } M(x, u_1, \ldots, u_i) = 1,$$

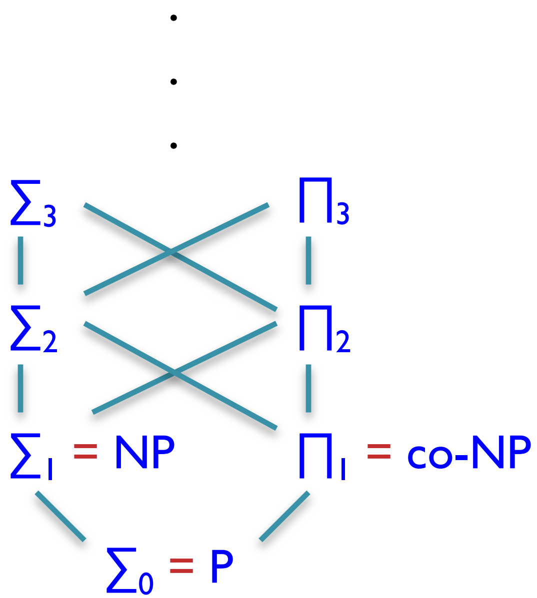
  where Q_i is $\forall$ or $\exists$ if i is odd or even, respectively.

- Obs. $\Sigma_i \subseteq \Pi_{i+1} \subseteq \Sigma_{i+2}$.

Recap: Polynomial Hierarchy

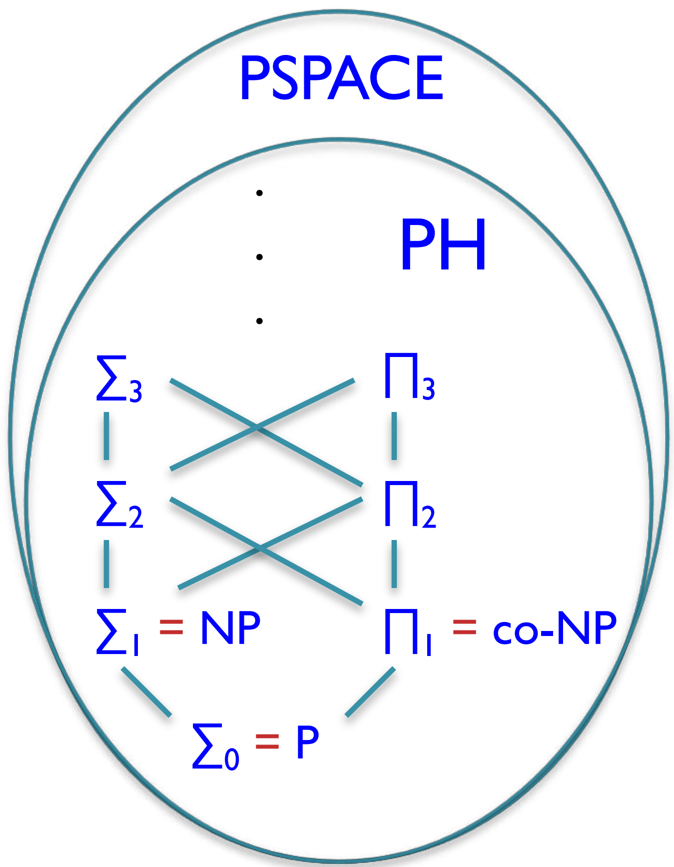
- Obs. $PH = \bigcup_{i \in \mathbf{N}} \Sigma_i = \bigcup_{i \in \mathbf{N}} \Pi_i$.

PH =



Recap: Polynomial Hierarchy

- Claim. PH ⊆ PSPACE .
- Proof. Similar to the proof of TQBF ∈ PSPACE.



Recap: Does PH collapse?

- General belief. Just as many of us believe $P \neq NP$ (i.e. $\Sigma_0 \neq \Sigma_1$) and $NP \neq co\text{-}NP$ (i.e. $\Sigma_1 \neq \Pi_1$), we also believe that for every i, $\Sigma_i \neq \Sigma_{i+1}$ and $\Sigma_i \neq \Pi_i$.

- Definition. We say PH collapses to the i-th level if $\Sigma_i = \Sigma_{i+1}$. (justified in the next theorem)

- Conjecture. There is no i such that PH collapses to the i-th level.

This is stronger than the $P \neq NP$ conjecture.

Recap: PH collapse theorems

- Theorem. If $\Sigma_i = \Sigma_{i+1}$ then $PH = \Sigma_i$.
- Theorem. If $\Sigma_i = \Pi_i$ then $PH = \Sigma_i$.

Recap: Complete problems in PH ?

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is P = PH ? …use poly-time Karp reduction!

- Definition. A language L' is *PH-hard* if for every L in PH, $L \leq_p L'$. Further, if L' is in PH then L' is *PH-complete*.

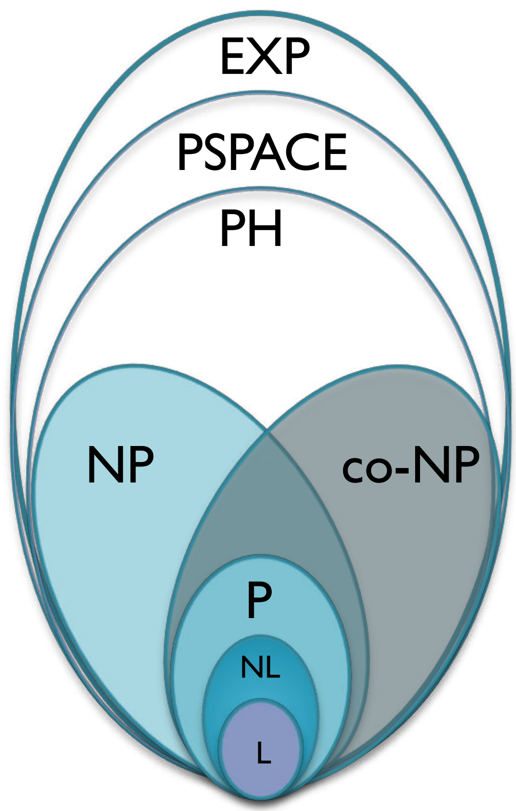
Recap: Complete problems in PH ?

- Fact. If L is poly-time reducible to a language in Σ_i then L is in Σ_i . (we've seen a similar fact for NP)

- Observation. If PH has a complete problem then PH collapses.

Recap: Complete problems in PH ?

- Fact. If L is poly-time reducible to a language in Σ_i then L is in Σ_i. (we've seen a similar fact for NP)
- Corollary. PH $\subsetneq$ PSPACE unless PH collapses.



Recap: Complete problems in Σ_i

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is $P = \Sigma_i$? …use poly-time Karp reduction!

- Definition. A language L' is Σ_i *-hard* if for every L in Σ_i , $L \leq_p L'$. Further, if L' is in Σ_i then L' is Σ_i *-complete*.

Recap: Complete problems in Σ_i

- **Definition.** The language Σ_i-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- **Theorem.** Σ_i-SAT is Σ_i *-complete*. (Σ_1-SAT is just SAT)

- **Observation.** Owing to the proof of the Cook-Levin theorem, we can assume that the formula in a Σ_i-SAT instance is a CNF (if i is odd) or a DNF (if i is even).

Recap: Other complete problems in Σ_2

- Ref. *"Completeness in the Polynomial-Time Hierarchy: A Compendium"* by *Schaefer and Umans (2008)*.

- Theorem. Eq-DNF and Succinct-SetCover are Σ_2 *-complete*.

An alternate characterization of PH

Oracle definition of Σ_i

- Definition. A language L is in $NP^{\Sigma_i\text{-SAT}}$ if there is a poly-time NTM with oracle access to Σ_i-SAT that decides L.

- Theorem. $\Sigma_{i+1} = NP^{\Sigma_i\text{-SAT}}$.

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- Theorem. $\Sigma_{i+1} = NP^{\Sigma_i\text{-SAT}}$.

- Observe that Σ_1-SAT = SAT. We'll prove the special case $\Sigma_2 = NP^{SAT}$. The proof of the theorem is similar.

Oracle definition of Σ_i

- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in Σ_2. There's a polynomial function q(.) and a poly-time TM M s.t.

$x \in L \iff \exists u \in \{0,1\}^{q(|x|)}\ \forall v \in \{0,1\}^{q(|x|)}$ s.t. $M(x,u,v) = 1$.

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- Proof. Let L be a language in Σ_2. There's a polynomial function q(.) and a poly-time TM M s.t.

$$x \in L \iff \exists u \in \{0,1\}^{q(|x|)}\ \forall v \in \{0,1\}^{q(|x|)} \text{ s.t. } \underbrace{\phi(x,u,v)}_{\text{Boolean circuit}} = 1.$$

Boolean circuit
(*by Cook-Levin*)

- In fact, owing to the proof of the Cook-Levin theorem, we can assume that ϕ is a DNF.

Oracle definition of Σ_i

- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in Σ_2. There's a polynomial function q(.) and a poly-time TM M s.t.

$x \in L \iff \exists u \in \{0,1\}^{q(|x|)}\ \forall v \in \{0,1\}^{q(|x|)}$ s.t. $\neg\phi(x,u,v) = 0$.

- Think of a NTM N that has the knowledge of M. On input x, it guesses $u \in \{0,1\}^{q(|x|)}$ non-deterministically and computes the circuit $\phi(x,u,v)$. Then, it queries the SAT oracle with $\neg\phi(x,u,v)$.

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- Note that $\neg\phi(x,u,v)$ is a CNF.

Oracle definition of Σ_i

- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
- Special case: N asks at most **one** query to the SAT oracle on every computation path on input x.

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- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
- Special case: N asks at most **one** query to the SAT oracle on every computation path on input x.
- We need to construct a Σ_2-statement that captures N's computation on input x.

Oracle definition of Σ_i

- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
- Special case: N asks at most one query to the SAT oracle on every computation path on input x.
- Think of a TM M that takes input x and $w \in \{0,1\}^{q(|x|)}$, $a_1 \in \{0,1\}$ and $u_1, v_1 \in \{0,1\}^{q(|x|)}$, where q(|x|) is the runtime of N on input x, and does the following:

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- M simulates N on input x with w as the non-deterministic choices.

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- Special case: N asks at most one query to the SAT oracle on every computation path on input x.
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- M simulates N on input x with w as the computation path. Suppose ϕ is the query asked by N on the path of computation defined by w.

Oracle definition of Σ_i

- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
- Special case: N asks at most one query to the SAT oracle on every computation path on input x.
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➢ If $a_1 = 1$ and $\phi(u_1) = 1$, M continues the simulation; else it stops and outputs 0. (In this case, M ignores v_1.)

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- Think of a TM M that takes input x and $w \in \{0,1\}^{q(|x|)}$, $a_1 \in \{0,1\}$ and u_1, $v_1 \in \{0,1\}^{q(|x|)}$, where $q(|x|)$ is the runtime of N on input x, and does the following:
- At the end of the simulation, M outputs whatever N outputs. Note: M is a poly-time TM.

Oracle definition of Σ_i

- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
- Special case: N asks at most one query to the SAT oracle on every computation path on input x.
- Observation. For any $w \in \{0,1\}^{q(|x|)}$ and $a_1 \in \{0,1\}$,

➢ N on computation path w gets answer a_1 from the SAT oracle and accepts x $\Leftrightarrow$

$\exists u_1 \in \{0,1\}^{q(|x|)}$ $\forall v_1 \in \{0,1\}^{q(|x|)}$ s.t. $M(x,w,a_1,u_1,v_1) = 1$.

(…will prove the observation shortly. Let's finish the proof.)

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- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
- Special case: N asks at most one query to the SAT oracle on every computation path on input x.
- $x \in L$ ⟺ $\exists w \in \{0,1\}^{q(|x|)}, a_1 \in \{0,1\}$ s.t

➢ N on computation path w gets answer a_1 from the SAT oracle and accepts x ⟺ $\exists w \in \{0,1\}^{q(|x|)}, a_1 \in \{0,1\}$

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- Special case: N asks at most one query to the SAT oracle on every computation path on input x.
- $x \in L \iff \exists w \in \{0,1\}^{q(|x|)}, a_1 \in \{0,1\}$ s.t
  - N on computation path w gets answer a_1 from the SAT oracle and accepts $x \iff \exists w \in \{0,1\}^{q(|x|)}, a_1 \in \{0,1\}$

$\exists u_1 \in \{0,1\}^{q(|x|)}\ \forall v_1 \in \{0,1\}^{q(|x|)}$ s.t. $M(x, \underbrace{w, a_1, u_1}_{\text{Call it } u}, v_1) = 1$.

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- Special case: N asks at most one query to the SAT oracle on every computation path on input x.
- $x \in L \Leftrightarrow \exists w \in \{0,1\}^{q(|x|)}, a_1 \in \{0,1\}$ s.t
- N on computation path w gets answer a_1 from the SAT oracle and accepts x $\Leftrightarrow$

  $\exists u \in \{0,1\}^{2q(|x|)+1}$ $\forall v_1 \in \{0,1\}^{q(|x|)}$ s.t. $M(x,u,v_1) = 1$.
- Therefore, L is in Σ_2 .

Proof of the observation

- Observation. For any $w \in \{0,1\}^{q(|x|)}$ and $a_1 \in \{0,1\}$,
- ➢ N on computation path w gets answer a_1 from the SAT oracle and accepts x ⟺

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- Proof.(⟹) M simulates N on computation path w. Let ϕ be the query asked by N to SAT.
- If $a_1 = 1$, $\exists u_1 \in \{0,1\}^{q(|x|)}$ $\phi(u_1) = 1$ and N accepts x.

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In this case, M ignores v_1.

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- Proof.(⇒) M simulates N on computation path w. Let ϕ be the query asked by N to SAT.
- If $a_1 = 0$, $\forall v_1 \in \{0,1\}^{q(|x|)}\ \phi(v_1) = 0$ and N accepts x.

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- Proof.(⟹) M simulates N on computation path w. Let ϕ be the query asked by N to SAT.
- Irrespective of the value of a_1,

  $\exists u_1 \in \{0,1\}^{q(|x|)} \ \forall v_1 \in \{0,1\}^{q(|x|)}$ s.t. $M(x,w,a_1,u_1,v_1) = 1$.

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- Proof.(⟸) Need to show that N on computation path w <u>gets answer a_1</u> from the SAT oracle. (*Homework*)

Oracle definition of Σ_i

- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
- General case: N asks at most q(|x|) queries to SAT oracle on every computation path on input x.
- *Homework*: Prove the general case. Define the poly-time machine M appropriately.

Oracles versus efficient algorithms

- Definition. A language L is in P^{SAT} if there is a poly-time TM with oracle access to SAT that decides L.
- $\Delta_2 := P^{SAT} \subseteq \Sigma_2 \cap \Pi_2$.
- A SAT oracle gives us the ability to solve SAT efficiently "much like" a poly-time algorithm for SAT.

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- Yet, in the first case we believe $P^{SAT} \neq NP^{SAT}$, whereas in the second case PH collapses to P, i.e., $P^{SAT} = NP^{SAT}$.
- Why? Think to understand the difference between oracles and poly-time algorithms for SAT (*Homework*).

Boolean Circuits

An algorithm for every input length?

- *"One might imagine that $P \neq NP$, but SAT is tractable in the following sense: for every ℓ there is a very short program that runs in time ℓ^2 and correctly treats all instances of size ℓ."* --- Karp and Lipton (1982).

An algorithm for every input length?

- *"One might imagine that P ≠ NP, but SAT is tractable in the following sense: for every ℓ there is a very short program that runs in time ℓ^2 and correctly treats all instances of size ℓ."* --- Karp and Lipton (1982).

- P ≠ NP rules out the existence of a single efficient algorithm for SAT that handles all input lengths. But, it doesn't rule out the possibility of having a sequence of efficient SAT algorithms – one for each input length.

Lesson learnt from Cook-Levin

- Locality of computation implies that an algorithm A working on inputs of some fixed length n and running in time $T(n)$ can be viewed as a Boolean circuit ϕ of size $O(T(n)^2)$ s.t. $A(x) = \phi(x)$ for every $x \in \{0,1\}^n$.
- On the other hand, a circuit on inputs of length n and of size S can be viewed as an algorithm working on length n inputs and running in time S.

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- On the other hand, a circuit on inputs of length n and of size S can be viewed as an algorithm working on length n inputs and running in time S.

- To rule the existence of a sequence of algorithms – one for each input length – we need to rule out the existence of a sequence of <u>(i.e., a family of) circuits</u>.

Boolean circuits

- A Boolean circuit is a directed acyclic graph whose nodes/gates are labelled as follows:
  - A node with in-degree zero is labelled by an input variable, and it outputs the value of the variable.
  - Any other node is labelled by one of the three operations $\wedge$, $\vee$, $\neg$, and it outputs the value of the operation on its input.

Nodes with out-degree zero are the output gates.

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- Typically, we'll consider circuits with one output gate, and with nodes having in-degree at most two.

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- **Size** of circuit is the no. of edges in it. **Depth** is the length of the longest path from an i/p to o/p node.

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θ(no. of nodes)

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  Nodes with out-degree zero are the output gates.

- **<u>Size</u>** corresponds to "sequential time complexity". **<u>Depth</u>** corresponds to "parallel time complexity".

Boolean circuits

- A <u>Boolean circuit</u> is a directed acyclic graph whose nodes/gates are labelled as follows:
  - A node with in-degree zero is labelled by an input variable, and it outputs the value of the variable.
  - Any other node is labelled by one of the three operations $\wedge$, $\vee$, $\neg$, and it outputs the value of the operation on its input.

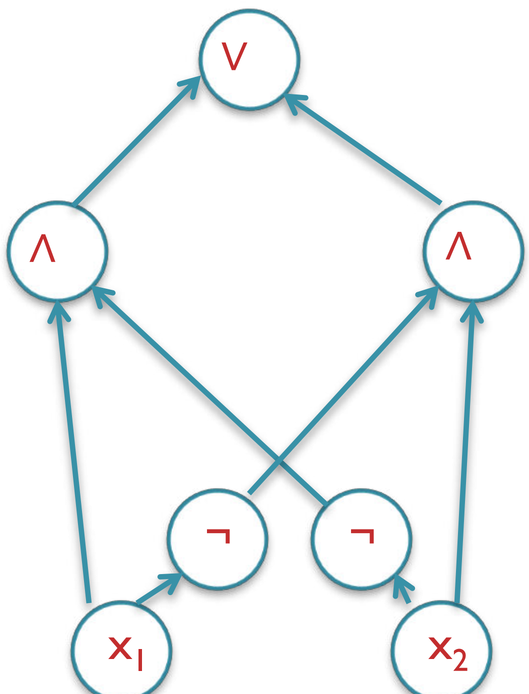
  Nodes with out-degree zero are the output gates.

- If every node in a circuit has out-degree at most one, then the circuit is called a <u>**formula**</u>.

A circuit for Parity

- $\text{PARITY}(x_1, x_2, \ldots, x_n) = x_1 \oplus x_2 \oplus \ldots \oplus x_n$.

$x_1 \oplus x_2 = (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2)$



ϕ

$\text{Size}(\phi) = |\phi| = 8$
$\text{Depth}(\phi) = 3$

Circuit family

- Let $T: \mathbb{N} \rightarrow \mathbb{N}$ be some function.
- Definition: A $T(n)$-size circuit family is a set of circuits $\{C_n\}_{n\in\mathbb{N}}$ such that C_n has n inputs and $|C_n| \leq T(n)$.

Class P/poly

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$$x \in L \Leftrightarrow C_n(x) = 1, \text{ where } n = |x|.$$

- Defintion: Class $P/poly = \bigcup_{c \geq 1} SIZE(n^c)$.

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The circuit family $\{C_n\}_{n \in N}$ *decides* L, i.e., C_n *decides* $L \cap \{0,1\}^n$.

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Alternatively, we say C_n *computes* the characteristic function of $L \cap \{0,1\}^n$.

- Defintion: Class $P/poly = \bigcup_{c \geq 1} SIZE(n^c)$.

Class P/poly

- Observation: $P \subseteq P/poly$.
- Proof. If $L \in P$, then there's a n^c-time TM that decides L for some constant c. By Cook-Levin, there's a $O(n^{2c})$-size circuit family $\{C_n\}_{n\in N}$ such that

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- Is P = P/poly? No! P/poly contains undecidable languages.

Class P/poly

- Let HALT = {(M,y) : M halts on input y}. HALT is an undecidable language.
- Notation. #(M,y) = number corresponding to the binary string (M,y).
- Let UHALT = $\{1^{\#(M,y)} : (M,y) \in \text{HALT}\}$. Then, UHALT is also an undecidable language.

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- Obs. Any unary language is in P/poly. (*Homework*)

  Hence, $\text{P} \subsetneq \text{P/poly}$.

Class P/poly

- What makes P/poly contain undecidable languages? *Ans:* $L \in$ P/poly implies that L is decided by a circuit family $\{C_n\}$, where $|C_n| = n^{O(1)}$. We don't require that C_n is poly-time computable from 1^n.

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- P/poly is a <u>*non-uniform class*</u> as a language in this class is allowed to have different algorithms/circuits for different input lengths.
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Model	What it captures
TM (uniform)	An algo for all inputs
Circuits (non-uniform)	An algo per i/p length

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- P/poly is a *non-uniform class* as a language in this class is allowed to have different algorithms/circuits for different input lengths.
- P is a *uniform class* as a language in this class has one algorithm for all inputs.
- Is SAT $\in$ P/poly? In other words, is NP $\subsetneq$ P/poly?

Karp-Lipton theorem

- Theorem (*Karp & Lipton 1982*). If NP $\subsetneq$ P/poly then PH = Σ_2.
- Proof. We'll show that NP $\subsetneq$ P/poly implies $\Pi_2 = \Sigma_2$. It's sufficient to show that $\Pi_2 \subseteq \Sigma_2$.

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- Theorem *(Karp & Lipton 1982)*. If NP $\subsetneq$ P/poly then PH = Σ_2 .
- Proof. Let L $\in \Pi_2$. There's a polynomial function q(.) and a poly-time TM M s.t.

  $x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)}\ \exists u_2 \in \{0,1\}^{q(|x|)}\ M(x,u_1,u_2) = 1$.

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- Goal. Come up with a polynomial function p(.) and a poly-time TM N s.t.

  $x \in L \Leftrightarrow \exists v_1 \in \{0,1\}^{p(|x|)}\ \forall v_2 \in \{0,1\}^{p(|x|)}\ N(x,v_1,v_2) = 1.$
- Think about designing such a TM N.

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by Cook-Levin

$$x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)} \exists u_2 \in \{0,1\}^{q(|x|)} \phi(x,u_1,u_2) = 1.$$

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- If M runs in time $T(n) = n^{O(1)}$ on (x,u_1,u_2), where $|x| = n$, then $|\phi| = O(T(n)^2)$. Let m = #(bits to write ϕ).
- N can compute ϕ from M in poly(|x|) time.

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$$x \in L \iff \forall u_1 \in \{0,1\}^{q(|x|)}\ \exists u_2 \in \{0,1\}^{q(|x|)}\ \phi(x,u_1,u_2) = 1.$$

$\phi(x,u_1,u_2)$ as a function of u_2 is satisfiable. Wlog ϕ is a CNF (why?).

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  $x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)}\ \phi(x,u_1, u_2) \in SAT.$

- By assumption, SAT $\in$ P/poly, i.e., there's a circuit C_m of size $p(m) = m^{O(1)}$ that correctly decides satifiability of all input circuits ϕ of length m.

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- First attempt. A Σ_2 statement to capture membership of strings in L.

$$x \in L \Leftrightarrow \exists C_m \in \{0,1\}^{p(m)} \; \forall u_1 \in \{0,1\}^{q(|x|)} \; C_m(\phi(x,u_1, u_2))=1.$$

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- Wrong! Think about a C_m that always outputs 1.

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- Need to be sure that C_m is the right circuit.

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- If there's a circuit C_m of size $m^{O(1)}$ that correctly decides satifiability of all input circuits ϕ of length m, then by self-reducibility of SAT, there's a multi-output circuit D_m of size $r(m) = m^{O(1)}$ that outputs a *satisfying assignment* for input ϕ if $\phi \in SAT$. *(Homework)*

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- A Σ_2 statement to capture membership in L.

$$x \in L \Leftrightarrow$$

$$\exists D_m \in \{0,1\}^{r(m)} \; \forall u_1 \in \{0,1\}^{q(|x|)} \;\; \phi(x,u_1, \underbrace{D_m(\phi(x,u_1,u_2))}_{\text{assignment to the } u_2 \text{ variables}}) = 1.$$

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Can be checked by a poly-time TM N.

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Karp-Lipton theorem

- Theorem (*Karp & Lipton 1982*). If NP $\subsetneq$ P/poly then PH $= \Sigma_2$.

- If we can show NP $\not\subset$ P/poly assuming P $\neq$ NP , then

  NP $\not\subset$ P/poly $\Leftrightarrow$ P $\neq$ NP .

- Karp-Lipton theorem shows NP $\not\subset$ P/poly assuming the stronger statement PH $\neq \Sigma_2$.