Computational Complexity Theory

Lecture 21: Complexity of Counting; 0/1-Perm is #P-complete

Department of Computer Science,
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Natural counting problems

- What is the complexity of the following problems?

- #SAT: Count the number of satisfying assignments of a given Boolean circuit/CNF.

- #HAMCYCLE: Count the number of Hamiltonian cycles in an undirected graph.

- Observation. The above problems are NP-hard.

Natural counting problems

- What is the complexity of the following problems?

- #PerfectMatching: Count the number of perfect matchings in a bipartite graph.

- #CYCLE: Count the number of simple cycles in a directed graph.

- Observation. The corresponding decision problems are in P.

Natural counting problems

- What is the complexity of the following problems?

- #PATH: Count the number of simple paths between two vertices in a connected graph.

- #SPANTREE: Count the number of spanning trees in a connected graph.

- Observation. The corresponding decision problems are trivial.

An easy counting problem

- Theorem. (*Kirchhoff 1847*) #SPANTREE is in FP.

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- Proof sketch. Let G be an n-vertex connected graph without self loops. Label the vertices by $\{1,\ldots, n\}$.

- Definition. The *Laplacian matrix* of G is an $n \times n$ matrix L_G defined as

$$
\begin{aligned}
L_G(i,j) &= \deg(i) && \text{if } i = j, \\
&= -1 && \text{if there's an edge } (i,j) \text{ in } G, \\
&= 0 && \text{otherwise.}
\end{aligned}
$$

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- Definition. The *Laplacian matrix* of G is an $n \times n$ matrix L_G defined as $L_G = D_G - A_G$, where D_G is the degree matrix and A_G the adjacency matrix of G.

- Observation. It is easy to compute L_G from A_G.

An easy counting problem

- Theorem. *(Kirchhoff 1847)* #SPANTREE is in FP.

- Proof sketch. Let G be an n-vertex connected graph without self loops. Label the vertices by $\{1,\ldots, n\}$.

- *Kirchhoff's matrix-tree theorem* states that

  no. of spanning trees of G = any cofactor of L_G.

- (i,j) *cofactor* of $L = (-1)^{i+j} \cdot \det($submatrix of L obtained by deleting the i-th row and the j-th column from $L)$.

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- Corollary. As determinant computation is in (functional) NC, #SPANTREES is in (functional) NC.

A hard counting problem

- Theorem. #CYCLE is in NP-hard.

- Lesson. A counting problem can be hard even if the corresponding decision problem is in P.

A hard counting problem

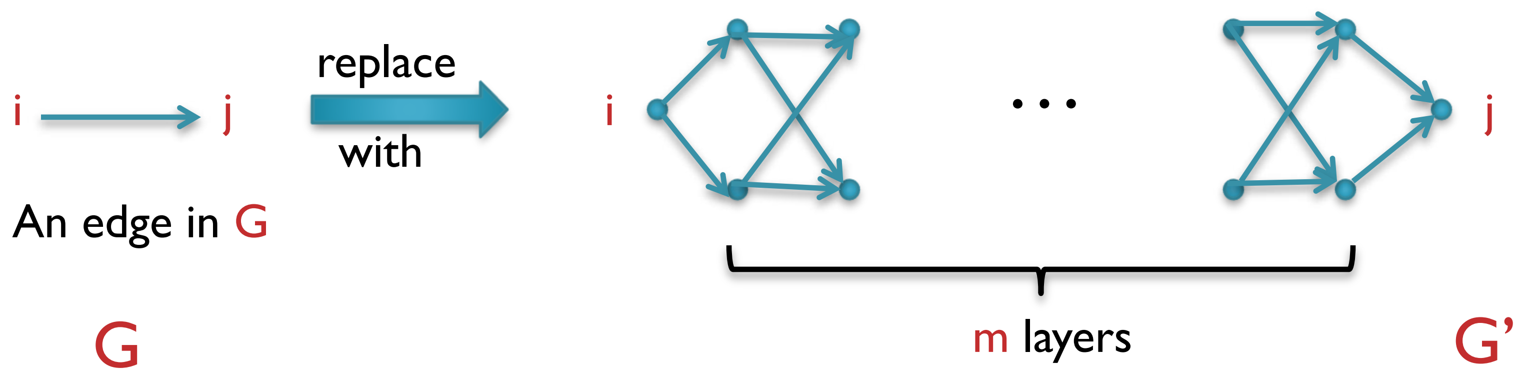
- Theorem. #CYCLE is in NP-hard.

- Proof. We will give a poly-time reduction from the Hamiltonian cycle problem to the #CYCLE problem.

A hard counting problem

- Theorem. #CYCLE is in NP-hard.

- Proof. Let G be an n-vertex digraph. We'll efficiently construct a new graph G' from G s.t. the presence of a Hamiltonian cycle in G can be readily derived from the number of cycles in G'. Construction of G' :

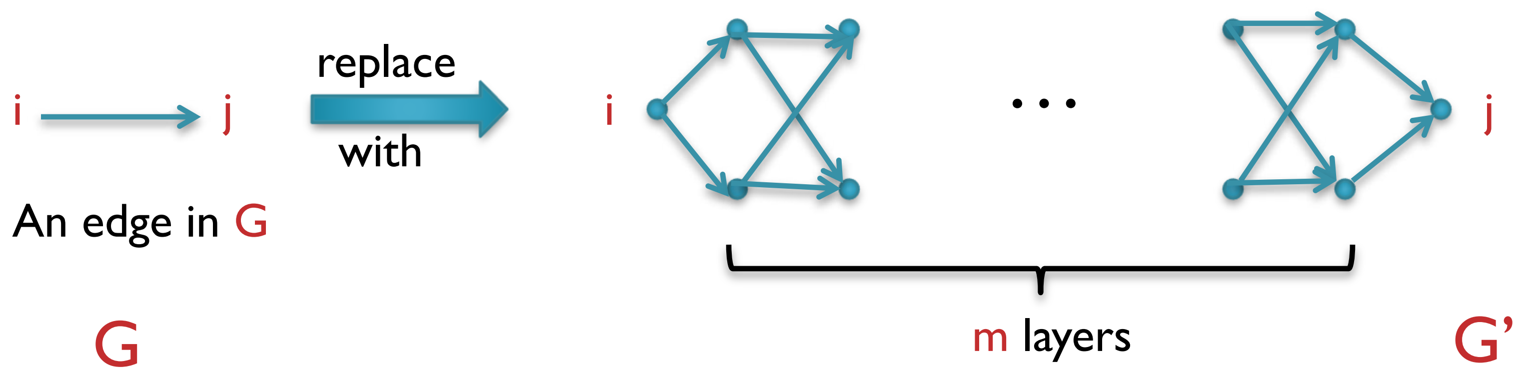


A hard counting problem

- Theorem. #CYCLE is in NP-hard.

- Proof. Case 1: If G has a HC, then #cycle(G') ≥ 2^{mn}.

#cycle → no. of cycles



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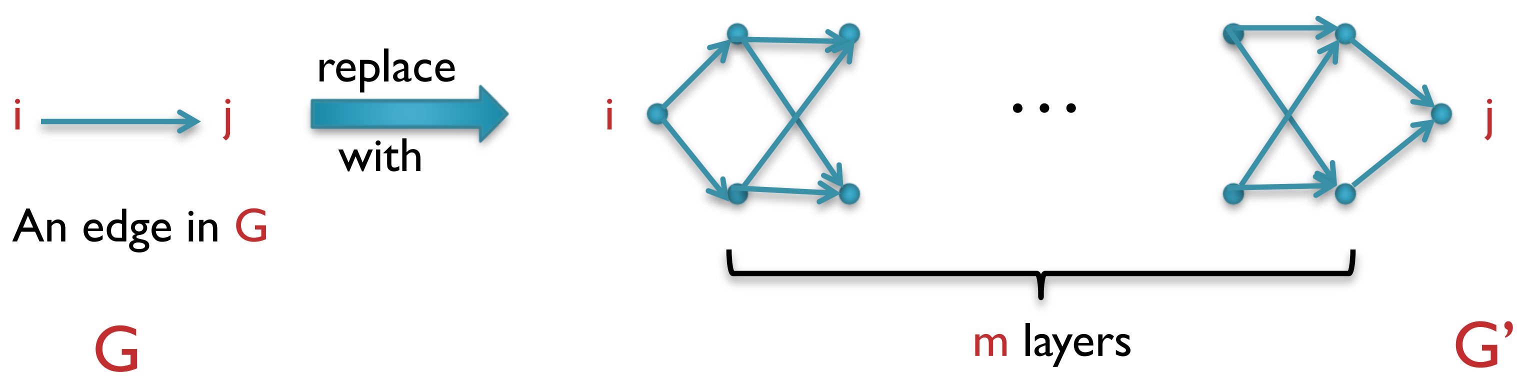

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- If we choose m such that $n^{n-1}.2^{m(n-1)} < 2^{mn}$, then we can find out if G has a HC from #cycle(G').
- Set $m = n^2$.

Class #P

- Definition. We say a function $f: \{0,1\}^* \to \mathbb{N}$ is in #P if there's a poly-time TM M and a polynomial function $p: \mathbb{N} \to \mathbb{N}$ such that for every $x \in \{0,1\}^*$,

$$f(x) = \left|\{u \in \{0,1\}^{p(|x|)} : M(x,u) = 1\}\right| .$$

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- Observation. Problems #SAT, #HAMCYCLE, #PerfectMatching, #CYCLE, #PATH and #SPANTREE are in #P.
- In fact, with every language in NP we can associate a counting problem that is in #P.

#P-completeness

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is #P = FP ?

#P-completeness

- Definition. A function $f: \{0,1\}^* \rightarrow N$ is in #P-complete if f is in #P and for every $g \in$ #P, we have $g \in FP^f$ i.e., g is poly-time Cook/Turing reducible to f.

- In other words, for every $x \in \{0,1\}^*$, we can compute $g(x)$ in polynomial time using oracle access to f.

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- Observation. If a #P-complete language is in FP then #P = FP.

Natural #P-complete problems

- Theorem. #SAT is #P-complete.

- Proof. #SAT is in #P. Let $g \in$ #P. We intend to show that $g \in \mathrm{FP}^{\#\mathrm{SAT}}$.

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- Algorithm: On input x, convert $M(x, ..)$ to a 3CNF ϕ_x using Cook-Levin theorem. Give ϕ_x as input to the #SAT oracle. Output whatever the oracle outputs.

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Note: Only one query to the oracle. Resembles a poly-time Karp reduction.

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$$g(x) = \left|\{u \in \{0,1\}^{p(|x|)} : M(x, u) = 1\}\right| .$$

- Correctness: Follows from the fact that the Cook-Levin reduction is *parsimonious*, i.e.,

$$\left|\{u \in \{0,1\}^{p(|x|)} : M(x, u) = 1\}\right| = \#\phi_x .$$

The no. of satisfying assignments of ϕ_x.

Natural #P-complete problems

- Theorem. #HAMCYCLE is #P-complete.

- Most (all?) NP-complete problems known till date have defining verifiers such that the corresponding counting problems are #P-complete.

- Open. Does every NP-complete problem have a defining verifier such that the corresponding counting problem is #P-complete ?

Issue: The reduction that shows NP-completeness of a problem needn't have to be parsimonious.

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- In fact, #PATH is #P-complete for both directed and undirected graphs.
- Theorem. *(Valiant 1979)* #PerfectMatching is #P-complete.
- Proof. We'll see a proof later.

Relation between #P and other classes

- Observation. #P $\subseteq$ PSPACE.

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Relation between #P and other classes

- Observation. $\#P \subseteq PSPACE$.
- Also, $PH \subseteq PSPACE$. How does #P relate to PH ?
- Theorem. *(Toda 1991)* $PH \subseteq P^{\#SAT}$.
- Hence, #P is *harder* than PH.

Approximations of #P functions

- Observation. If #P = FP, then P = NP.
- Open. Does P = NP imply #P = FP ?

- But, we do know that P = NP implies every #P problem has a randomized polynomial-time *approximation* algorithm.

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- But, we do know that P = NP implies every #P problem has a <u>randomized</u> polynomial-time approximation algorithm.

Can be derandomized!

Approximations of #P functions

- Definition. A function $f: \{0,1\}^* \rightarrow \mathbb{N}$ has a *Fully Polynomial-time Randomized Approximation Scheme* (FPRAS) if for every $\varepsilon, \delta > 0$, there's a PTM M such that for every $x \in \{0,1\}^*$,
  - $(1-\varepsilon).f(x) \leq M(x) \leq (1+\varepsilon).f(x)$ with prob. $\geq 1 - \delta$,
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- Theorem. If $P = NP$ then every #P function has a FPRAS.
- Remark. In fact the above FPRAS can be replaced by a FPTAS (*Fully Poly-Time Approximation Scheme*).

Approximations of #P functions

- Some #P-complete problems do admit FPRAS unconditionally!

- Theorem. *(Jerrum, Sinclair, Vigoda 2001)* #PerfectMatching has a FPRAS.

- Remark. No derandomization of this algorithm is known!

Approximations of #P functions

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- Theorem. *(Jerrum, Sinclair, Vigoda 2001)* Permanent of a square matrix with non-negative entries has a FPRAS.

- If $X = (x_{ij})_{i,j\in n}$ then $\mathrm{Perm}(X) = \sum_{\sigma\in S_n} \prod_{i\in[n]} x_{i\,\sigma(i)}$.

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- Note. If B_G is the biadjacency matrix of a bipartite graph G, then $\mathrm{Perm}(B_G)$ = #PerfectMatching(G).

  ↑ 0/1 matrix

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- Theorem. *(Valiant 1979)* 0/1-Perm is #P-complete.

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- Proof. 0/1-Perm is in #P. (Why?)

0/1-Permanent is #P-complete

- Theorem. *(Valiant 1979)* 0/1-Perm is #P-complete.
- Proof. We'll show that #3SAT $\in$ $FP^{0/1\text{-Perm}}$.
- In fact, we'll give a poly-time "Karp-like" reduction from #3SAT to 0/1-Perm, i.e., we'll give a poly-time computable function that maps a 3CNF ϕ to a 0/1-matrix A_ϕ s.t. $\#\phi$ is efficiently computable from $\text{Perm}(A_\phi)$
- This means only <u>one query</u> to the 0/1-Perm oracle is required.

Graph theoretic interpretation of Perm

- Let $A = (a_{ij})_{i,j\in r}$, where $a_{ij} \in R$.
- Then, $\mathrm{Perm}(A) = \sum_{\sigma\in S_r} \prod_{i\in[r]} a_{i\,\sigma(i)}$.
- Let G be the weighted digraph on r vertices with adjacency matrix A, i.e., the edge (i, j) in G has weight a_{ij}.

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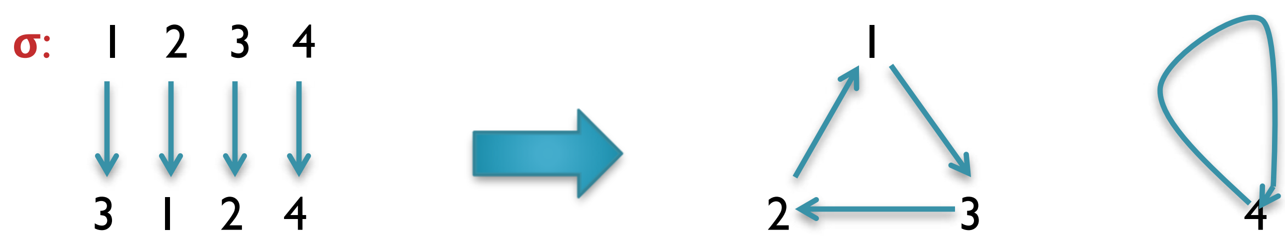
- Every permutation σ: $[r] \rightarrow [r]$ can be expressed (uniquely) as a product of disjoint cycles.



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- Definition. A *cycle cover* of a digraph G is a subgraph of G having in-degree and out-degree of every vertex exactly 1, i.e., the subgraph is a disjoint union of cycles covering all the vertices of G.

- *Weight* of a cycle cover C, denoted wt(C), is defined as the product of the weights of the edges in C.

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- Observation. $\text{Perm}(A) = \sum_{\substack{C:\ C \text{ is cycle} \\ \text{cover of } G}} \text{wt}(C)$.

Every "contributing" permutation σ corresponds to a cycle cover C and vice versa.

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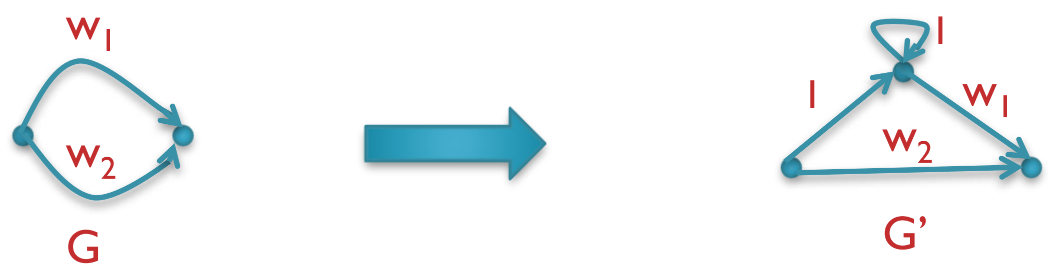
Graph with parallel edges

- Note. We can talk about "adjacency matrix" of a graph G that has parallel edges by defining a new graph G':



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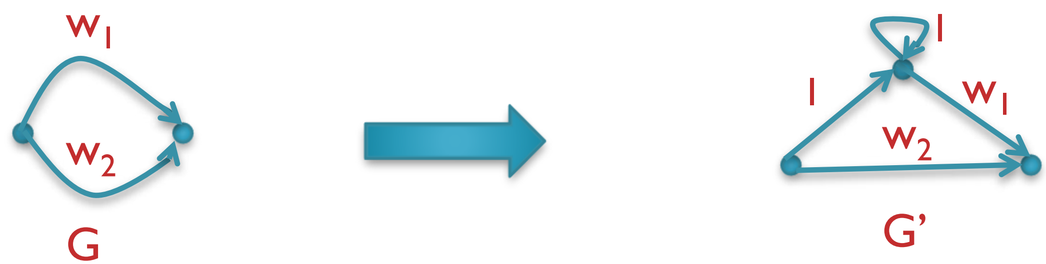

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- Observation. $$\sum_{C:\ C \text{ is cycle cover of } G} wt(C) = \sum_{C:\ C \text{ is cycle cover of } G'} wt(C).$$

0/1-Permanent is #P-complete

- Theorem. *(Valiant 1979)* 0/1-Perm is #P-complete.
- Proof. Let ϕ be a 3CNF that has n variables and m clauses. Assume that every clause has <u>exactly</u> 3 literals.

- Step 1: From ϕ we'll form a graph $H = H_\phi$ that has edge weights in $\{-1, 0, 1, 2, 3\}$ such that

$$\mathrm{Perm}(A_H) = \sum_{C:\ C \text{ is cycle cover of } H} \mathrm{wt}(C) = 4^{3m}.\#\phi \,. \qquad \text{... Eqn (1)}$$

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- Note. Eqn (1) doesn't give a FPRAS for #3SAT as the FPRAS for Perm is for matrices with *non-negative entries*.

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- Proof. Let ϕ be a 3CNF that has n variables and m clauses. Assume that every clause has <u>exactly</u> 3 literals.

- Step 2: We'll process H further to get a new graph $G = G_\phi$ with edge weights in {0,1} such that $\#\phi$ can be efficiently computed from $\mathrm{Perm}(A_G)$.

- However, unlike Eqn (1), we won't get an "precise" equation relating $\mathrm{Perm}(A_G)$ and $\#\phi$.

Details of Step 1 and Step 2

Step 1: Construction of H

- Convention. In the figures, edges without labels have weight 1, and missing edges have weight 0.

- H will be constructed using 3 kinds of *gadgets* (graphs):

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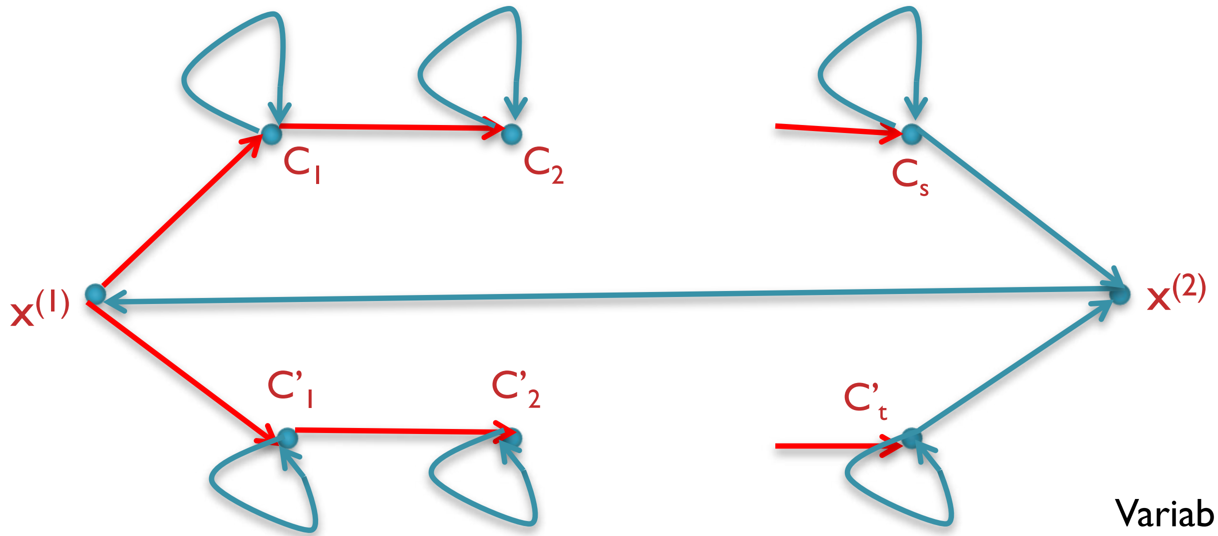
- H will be constructed using 3 kinds of <u>*gadgets*</u> (graphs):
  - Variable gadgets (there will be n of them),
  - Clause gadgets (there will be m of them), and
  - XOR gadgets.

- XOR gadgets are cleverly constructed 4-vertex graphs which will be used to connect variable gadgets with clause gadgets.

A variable gadget

- Let x be a variable. $C_1, \ldots, C_s$ be the clauses in which x appears, and $C'_1, \ldots, C'_t$ the clauses in which ¬x appears.



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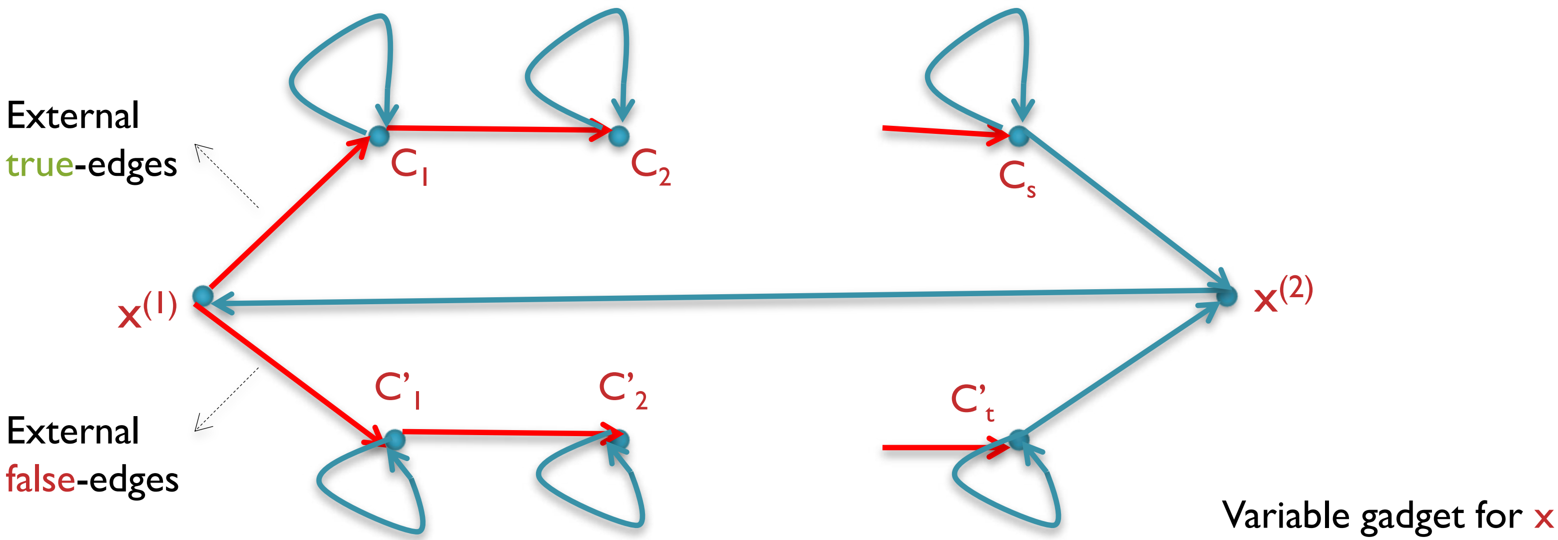

- The external edges (i.e., the red edges) will <u>not</u> be present in H, they will be used to connect to the Clause gadgets via the XOR gadgets.

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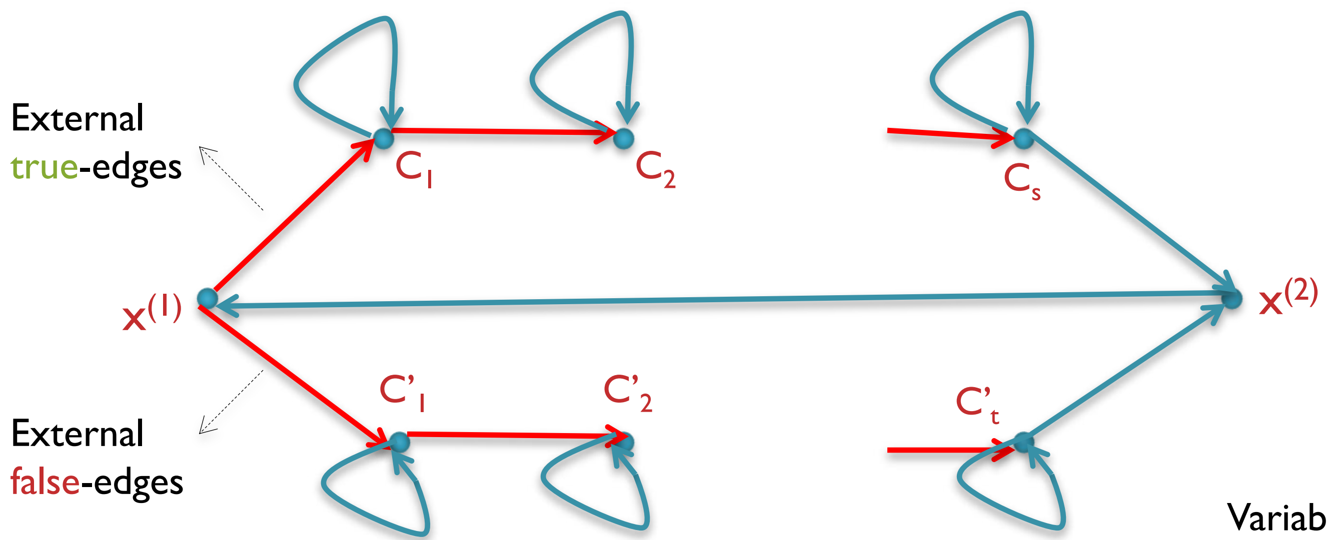
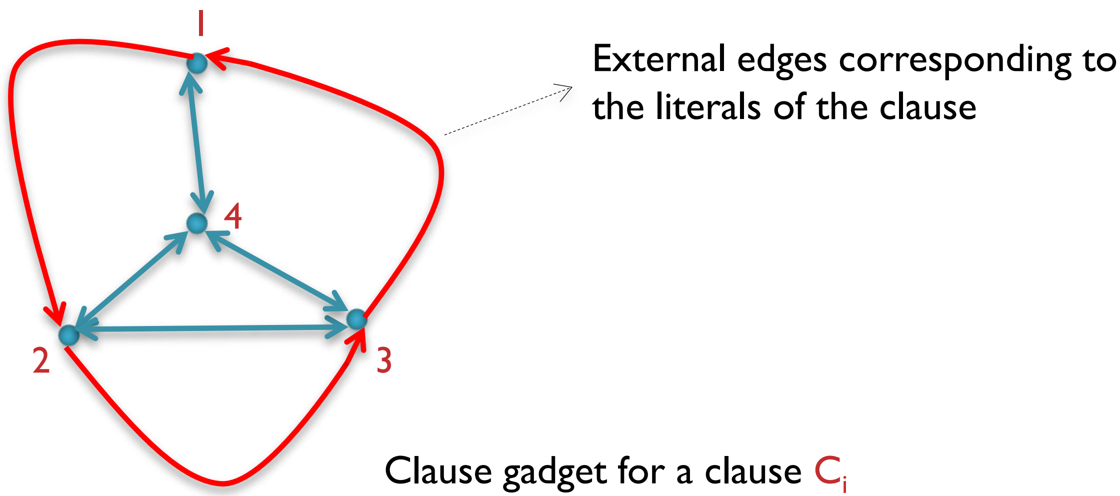


Variable gadget for x

- Observation 1. A variable gadget has exactly 2 cycle covers corresponding to 0/1 assignment to the variable.

A clause gadget

- Has 4 vertices and 3 external edges (i.e., red edges) corresponding to the 3 literals of the clause.



Clause gadget for a clause C_i

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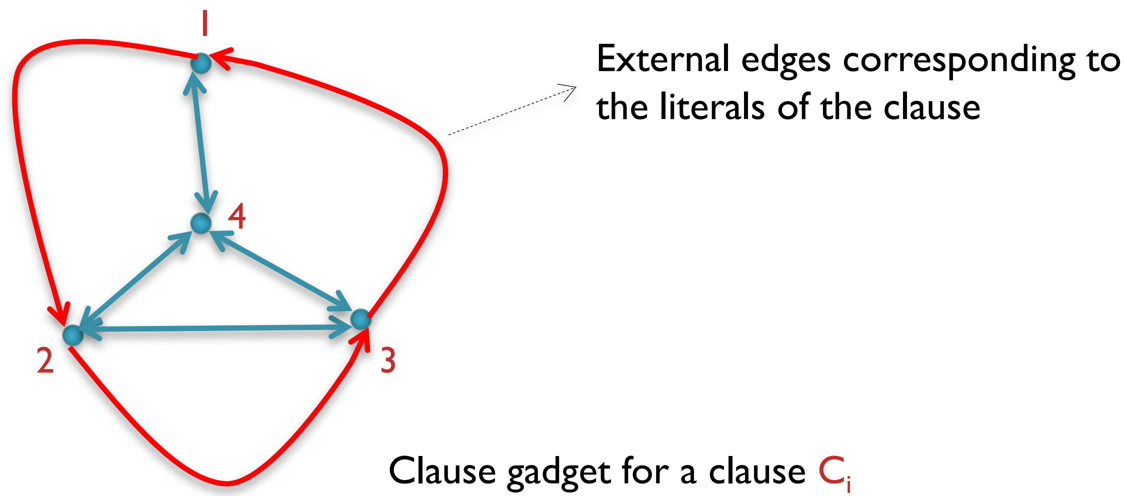

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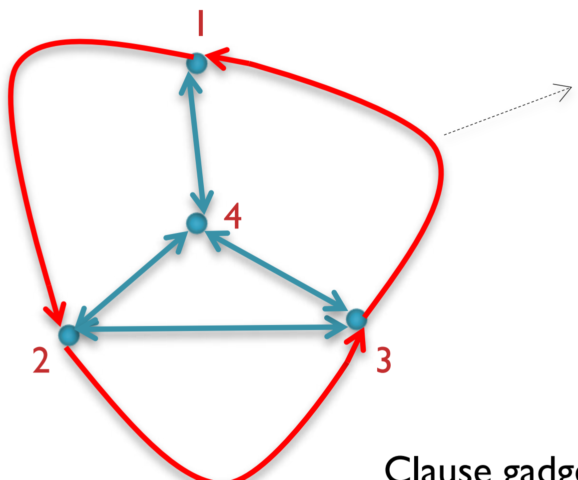


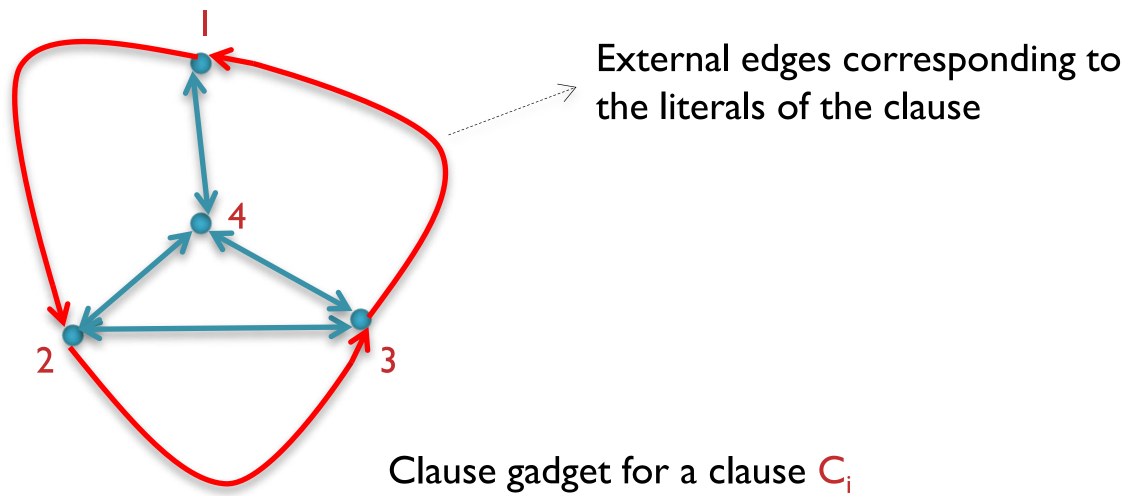
Clause gadget for a clause C_i

- Observation 2a. The only possible cycle covers of a clause gadget are those that exclude at least one external edge.

Excluding an external edge will indicate that the corresponding literal is set to 1.

A clause gadget

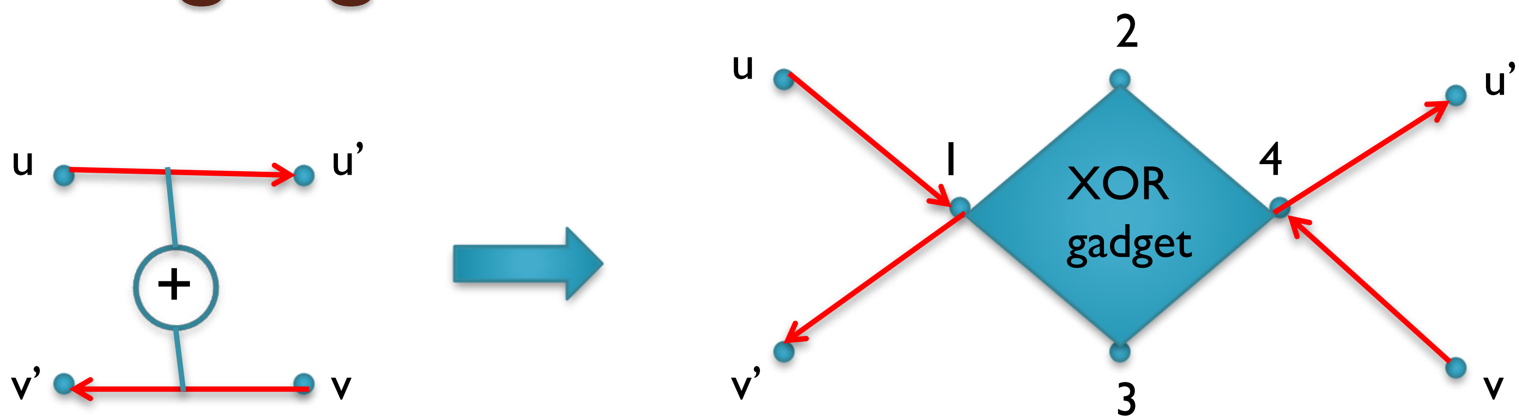
- Has 4 vertices and 3 external edges (i.e., red edges) corresponding to the 3 literals of the clause.



Clause gadget for a clause C_i

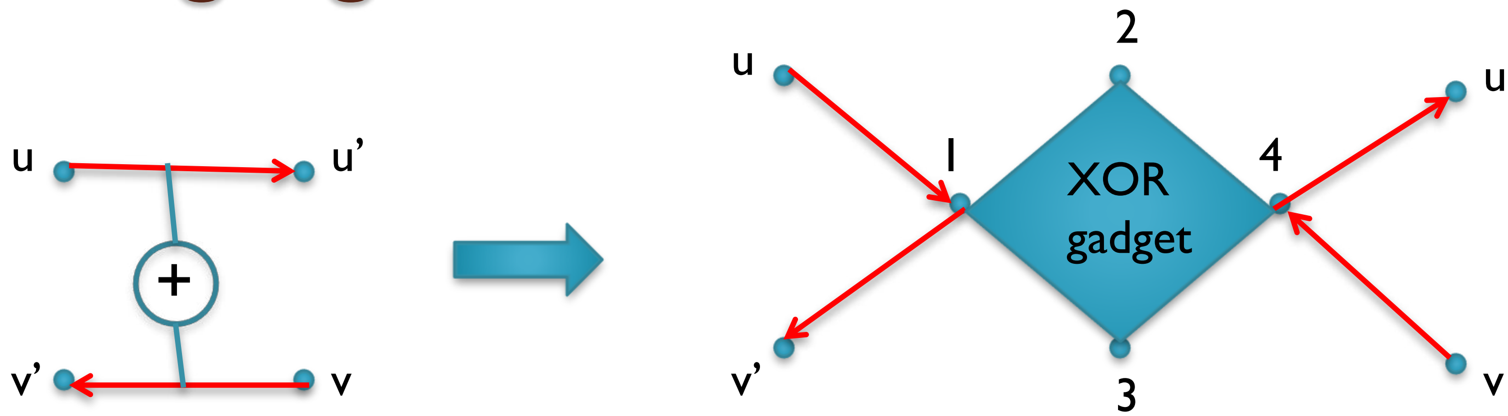
- Observation 2b. For any given proper subset of the 3 external edges, there's a unique cycle cover (of weight 1) that contains them.

XOR gadget



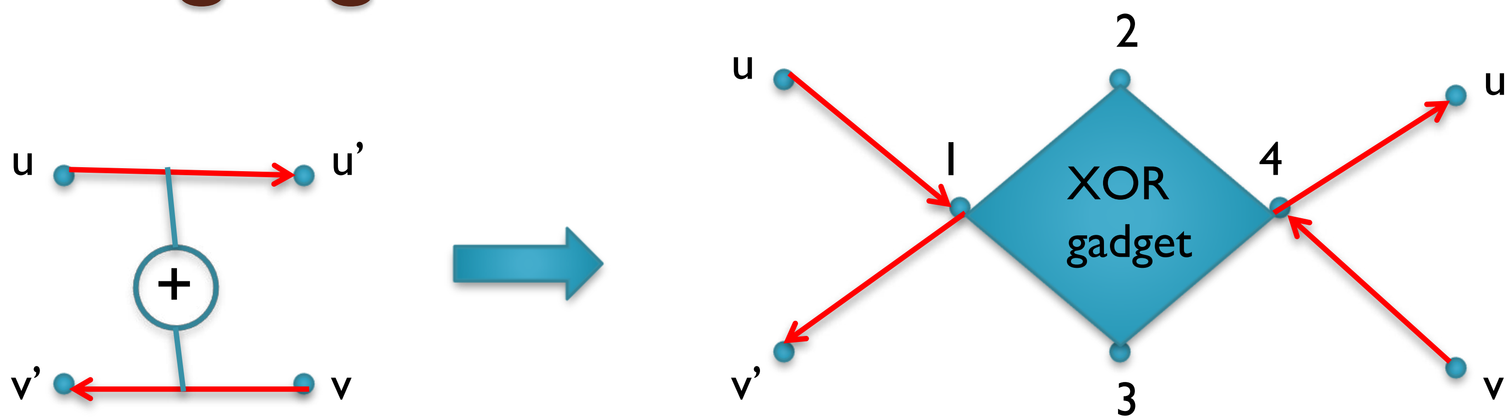
- We'll construct an XOR gadget such that the following features are satisfied:

XOR gadget



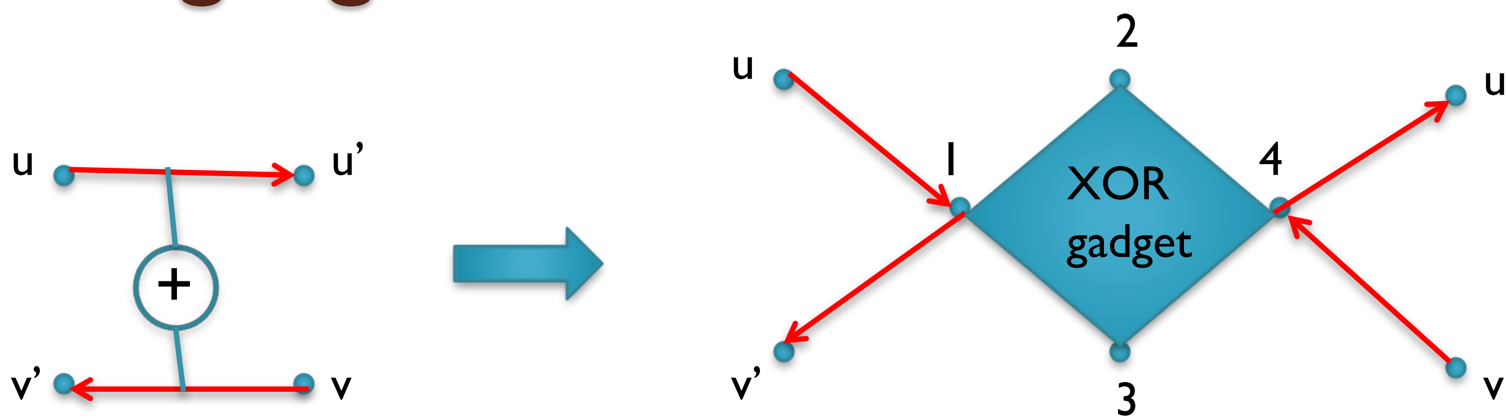
- We'll construct an XOR gadget such that the following features are satisfied:
  - Feature 1: Consider cycle covers of H that contain a <u>fixed</u> set of edges outside the XOR gadget but contain <u>none</u> of (u,1), (1,v'), (v,4), (4,u'). The sum of the weights of all such cycle covers is 0.

XOR gadget



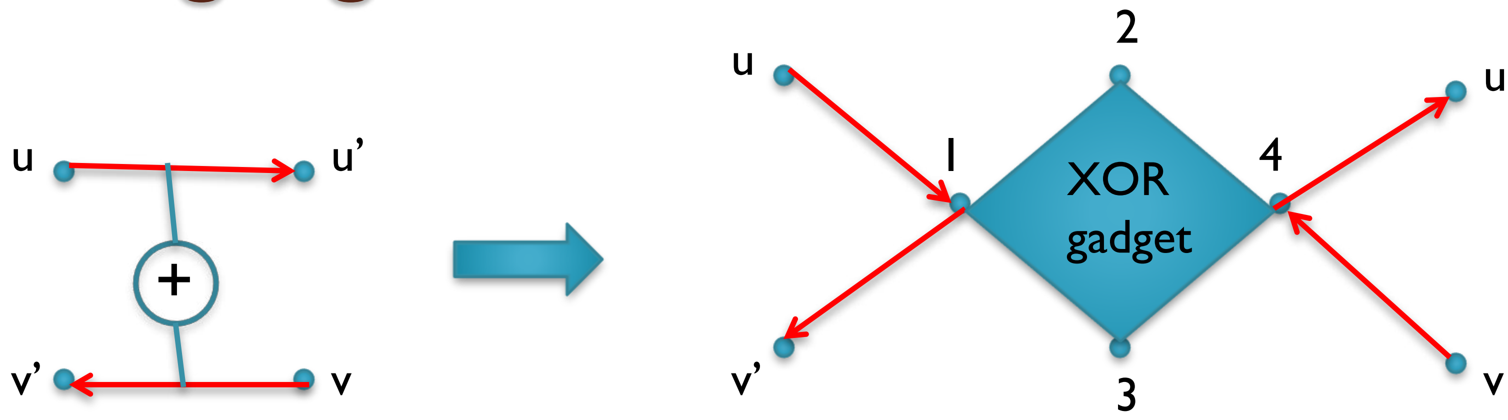
- We'll construct an XOR gadget such that the following features are satisfied:
  - Feature 2: Consider cycle covers of H that contain a <u>fixed</u> set of edges outside the XOR gadget including <u>*at least one*</u> of the pairs ((u,1), (1,v')) and ((v,4), (4,u')). The sum of the weights of all such cycle covers is 0.

XOR gadget



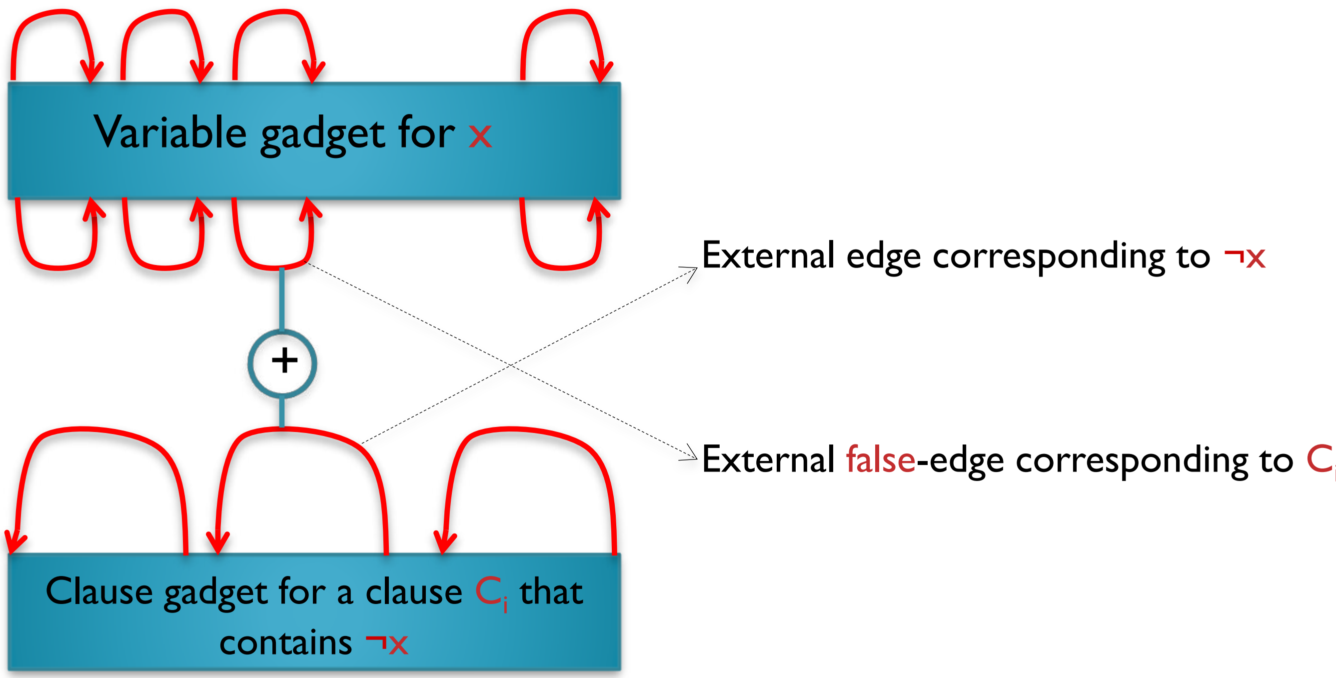
- We'll construct an XOR gadget such that the following features are satisfied:
  - Feature 3: Consider cycle covers of H that contain a fixed set of edges outside the XOR gadget including (u,1), (4,u') but not (v,4), (1,v'). The sum of the weights of all such cycle covers is 4.(product of the weights of the fixed set of edges).

XOR gadget



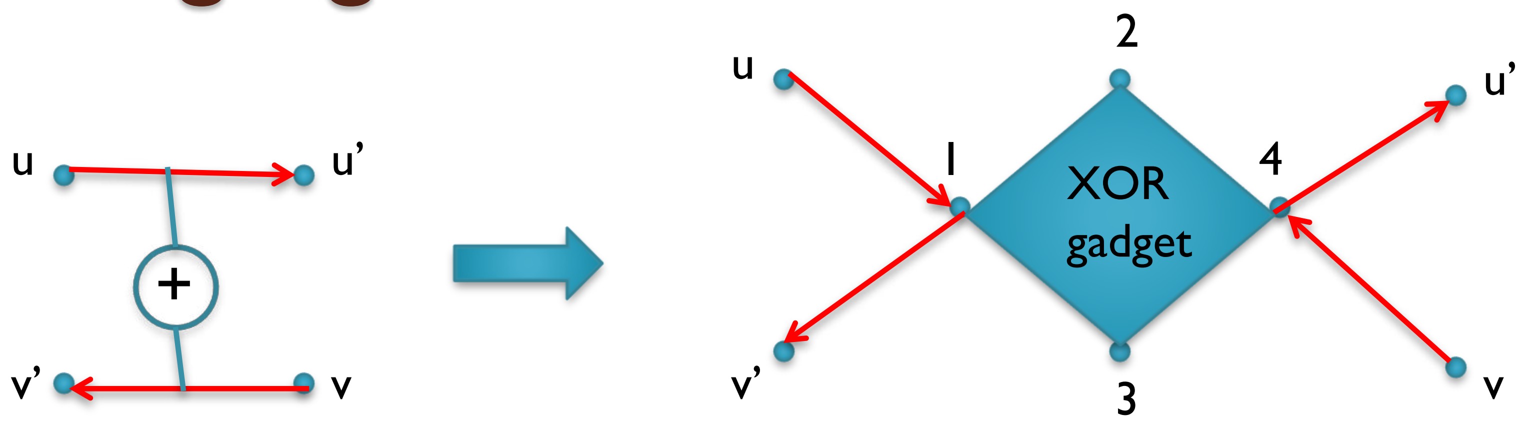
- We'll construct an XOR gadget such that the following features are satisfied:
  - ➢ Feature 4: Consider cycle covers of H that contain a <u>fixed</u> set of edges outside the XOR gadget including (v,4), (1,v') but not (u,1), (4,u'). The sum of the weights of all such cycle covers is 4.(product of the weights of the <u>fixed</u> set of edges).

Construction of H



- Size(H) = poly(n,m).
- There are 3m XOR gadgets in H. Every cycle cover of H "touches" the 3m XOR gadgets.

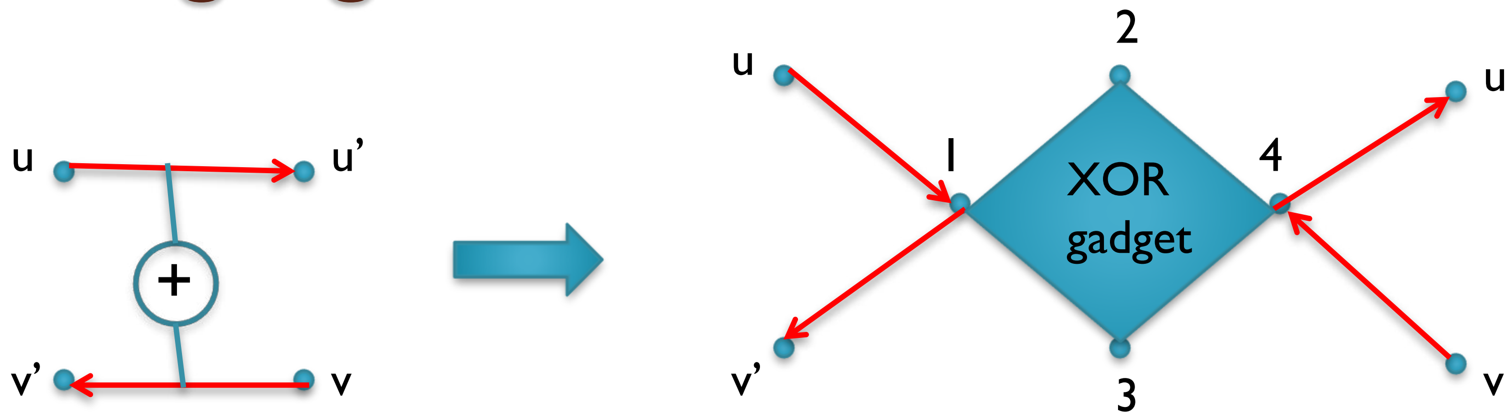
XOR gadget



- An XOR gadget can be "touched" in 4 possible ways:
  a. None of (u,1), (1,v'), (v,4), (4,u'),
  b. At least one of the pairs ((u,1),(1,v')) & ((v,4),(4,u')),
  c. Only (u,1), (4,u'),
  d. Only (v,4), (1,v').

Call these the "<u>touching patterns</u>" of an XOR gadget.

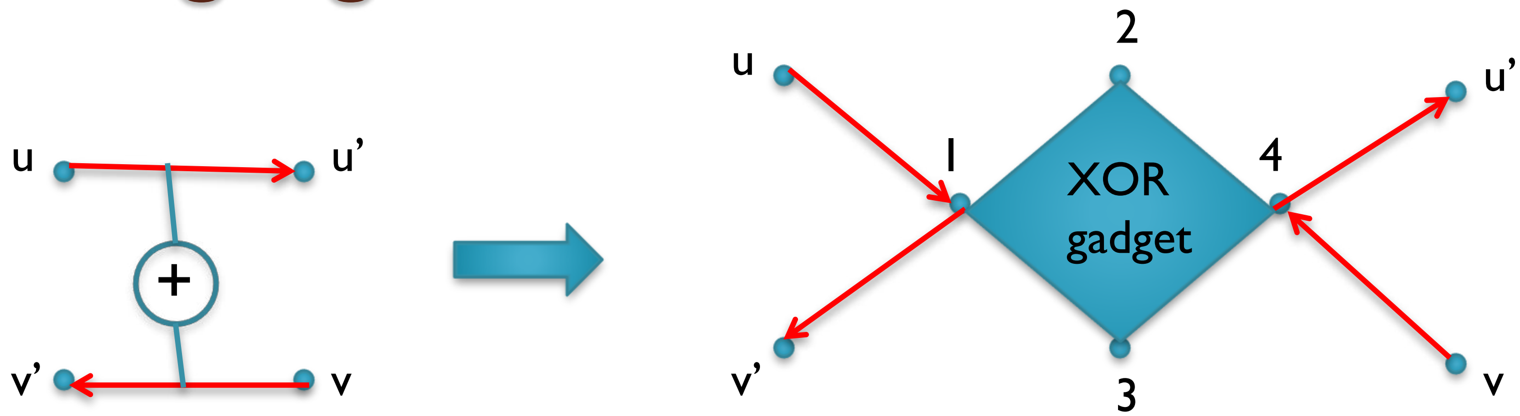
XOR gadget



- Every cycle cover of H can be mapped to a specific choice of the "touching patterns" of the 3m XOR gadgets.

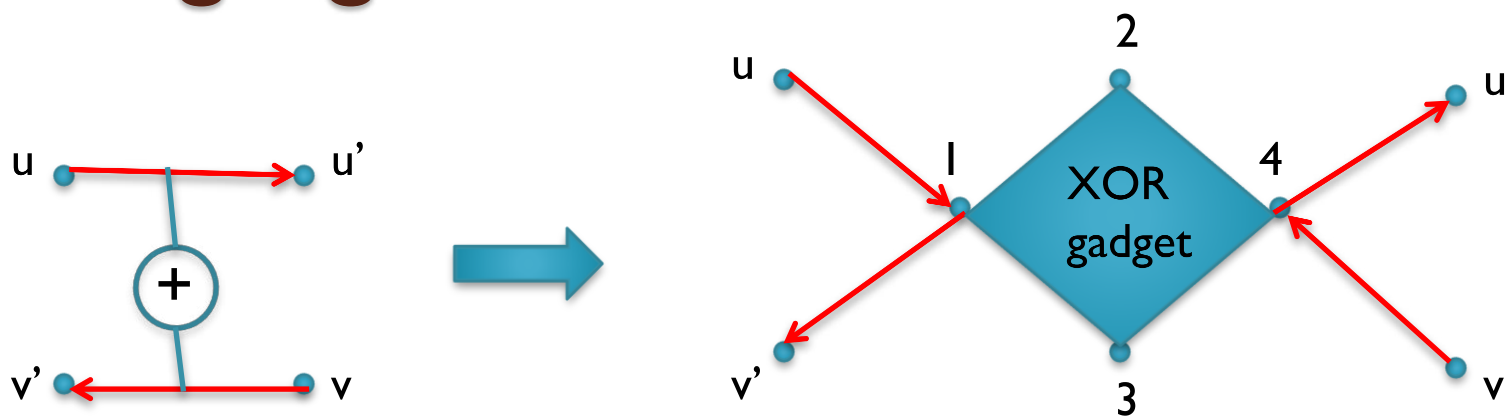
- Now, let us examine the sum of the weights of all the cycle covers of H.

XOR gadget



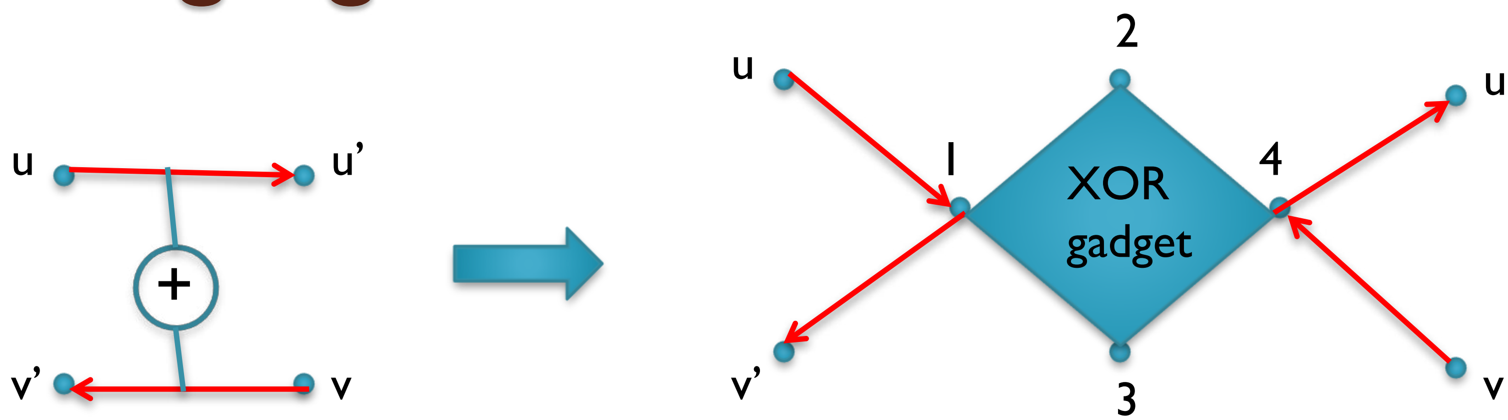
- Claim 1a. Cycle covers, which map to a <u>specific</u> choice of the "touching patterns" of the XOR gadgets s.t. the "touching pattern" of <u>at least one</u> of the XOR gates is of type a, <u>**do not**</u> contribute to the final sum.
- Proof. Follows from Feature 1. *(Homework)*

XOR gadget



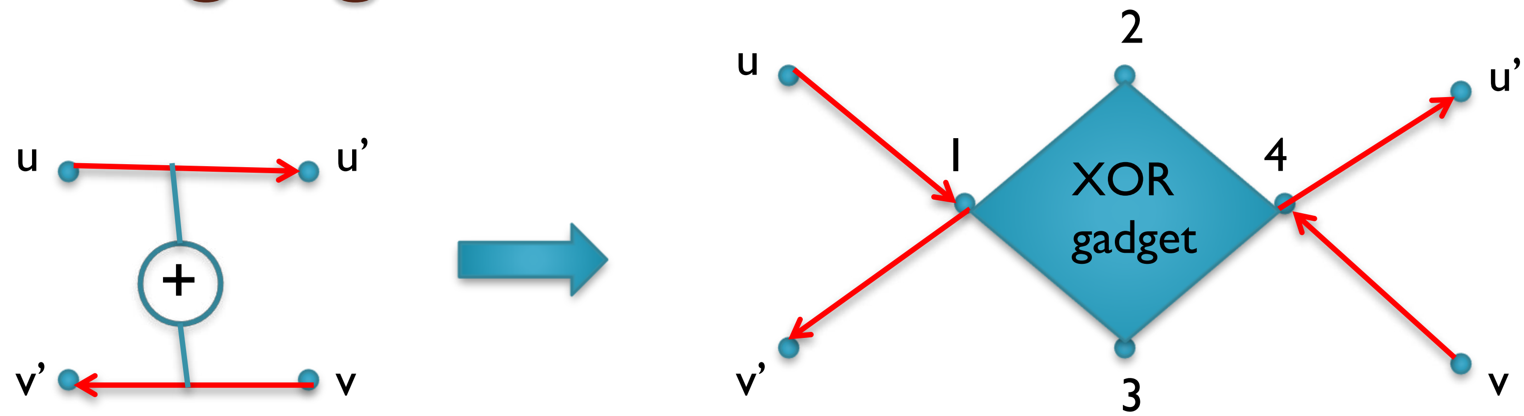
- Claim 1b. Cycle covers, which map to a <u>specific</u> choice of the "touching patterns" of the XOR gadgets s.t. the "touching pattern" of <u>at least one</u> of the XOR gates is of type b, <u>**do not**</u> contribute to the final sum.
- Proof. Follows from Feature 2. *(Homework)*

XOR gadget



- Claim 1c. Cycle covers, which map to a specific choice of the "touching patterns" of the XOR gadgets s.t. the "touching pattern" of every XOR gate is of type c or d, together contribute 4^{3m} or 0 to the final sum.
- Proof. Follows from Feature 3 & 4, and Observations 2a, 2b & 1. *(Homework)*

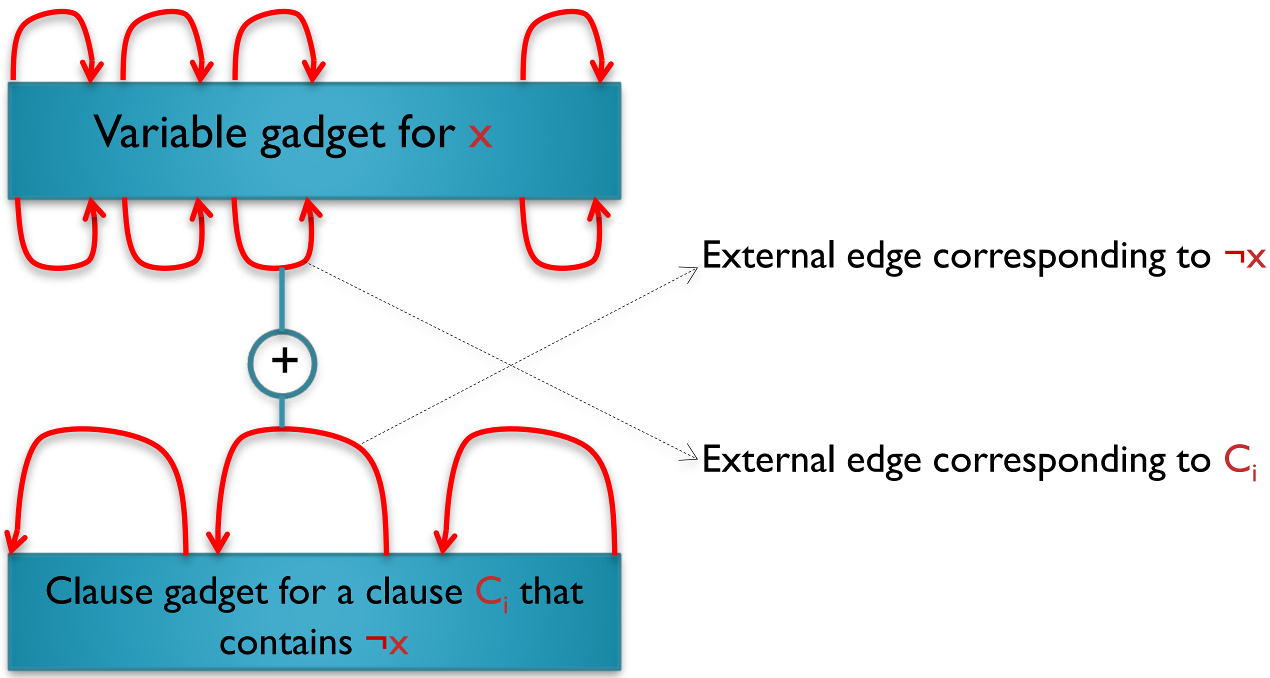
XOR gadget



- Claim 1a, 1b and 1c justify the name of the "XOR" gadget.

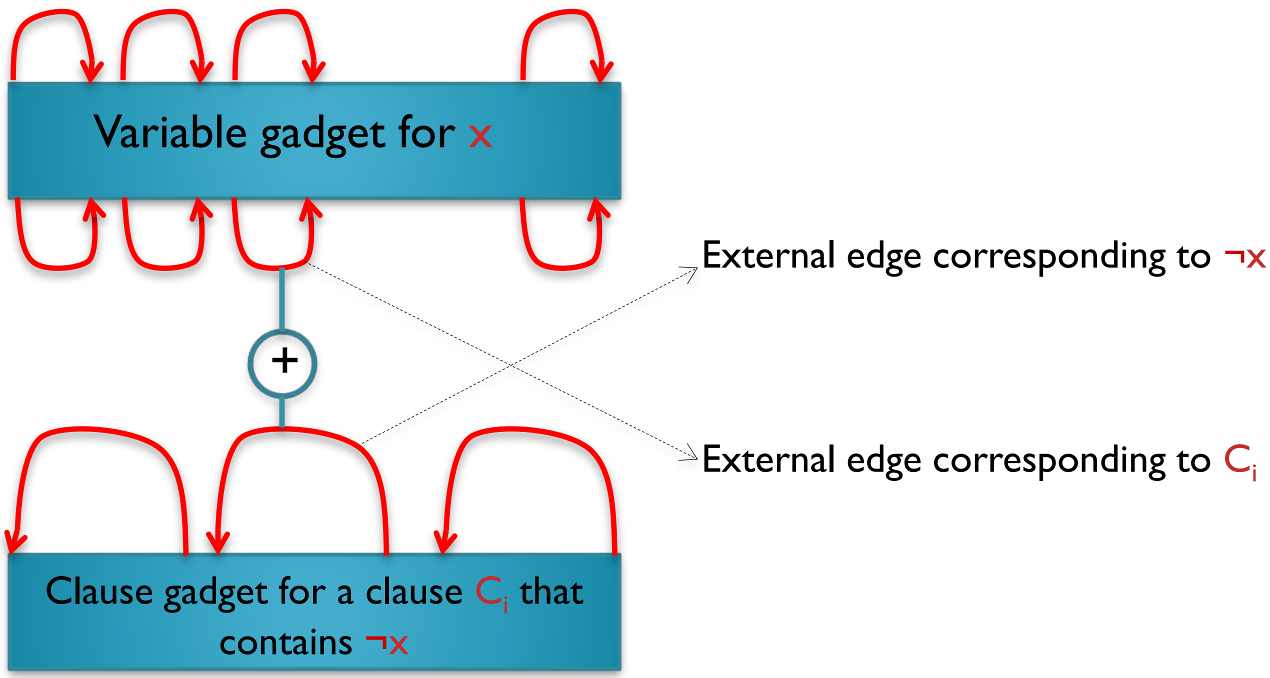
- The XOR gadget ensures that either the "edge" (u,u') or the "edge" (v,v') is taken in a potentially contributing choice of the "touching patterns" of the XOR gadgets.

Construction of H



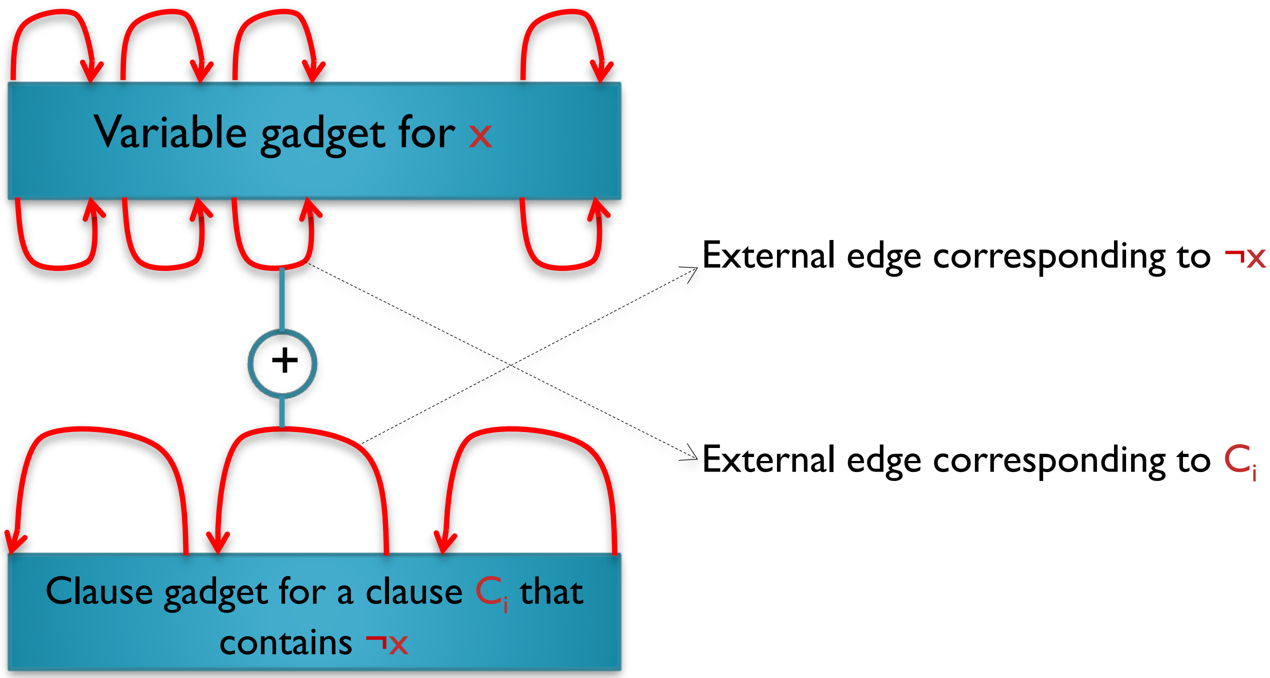
- Observation 3. Every <u>potentially</u> contributing choice of the "touching patterns" of the XOR gadgets can be mapped to a <u>unique</u> choice of the cycle covers of the variable gadgets. *(Homework)*

Construction of H



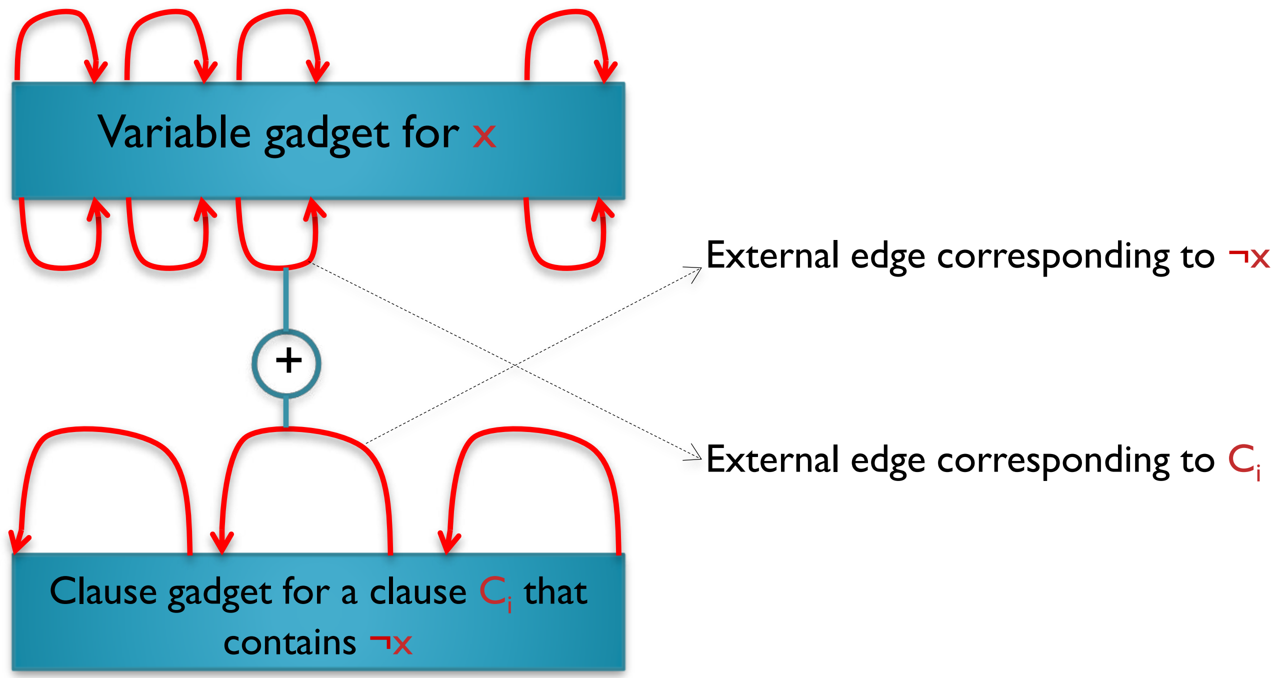
- Recall (from Observation 1) that a variable gadget has exactly 2 cycle covers corresponding to 0/1 assignment to the variable.

Construction of H



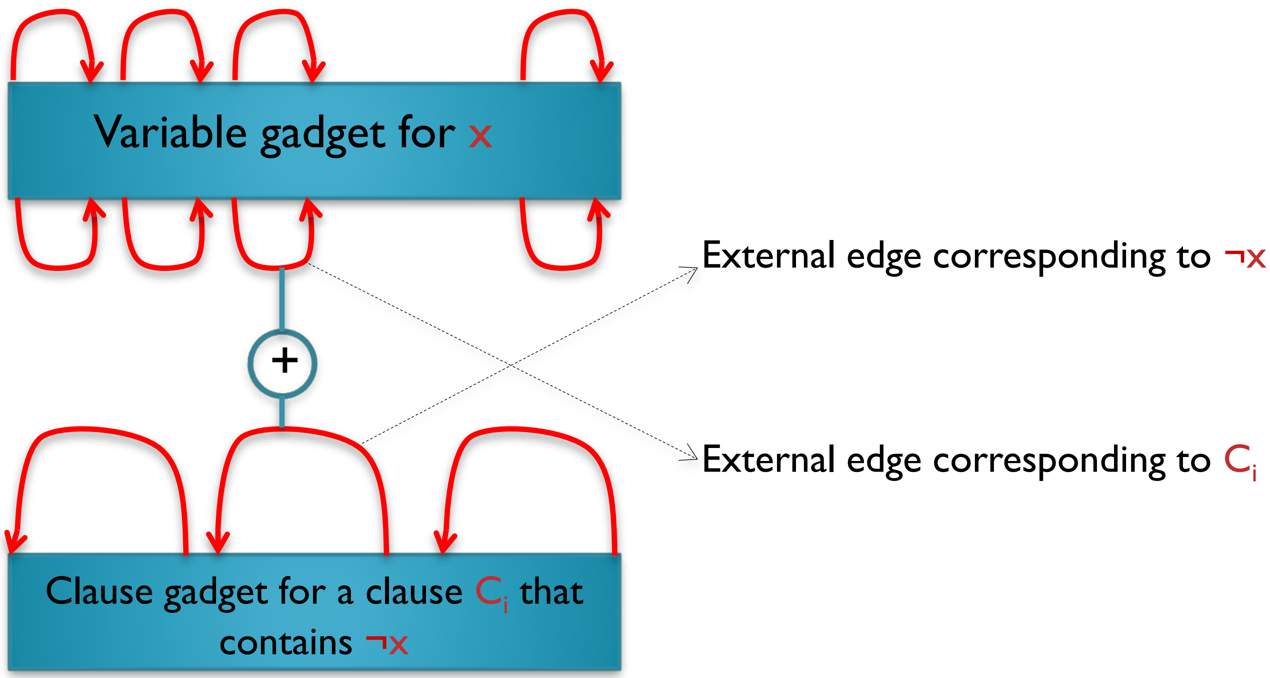
- Observation 3. (put differently) Every potentially contributing choice of the "touching patterns" of the XOR gadgets can be mapped to a unique 0/1 assignment to the variables.

Construction of H



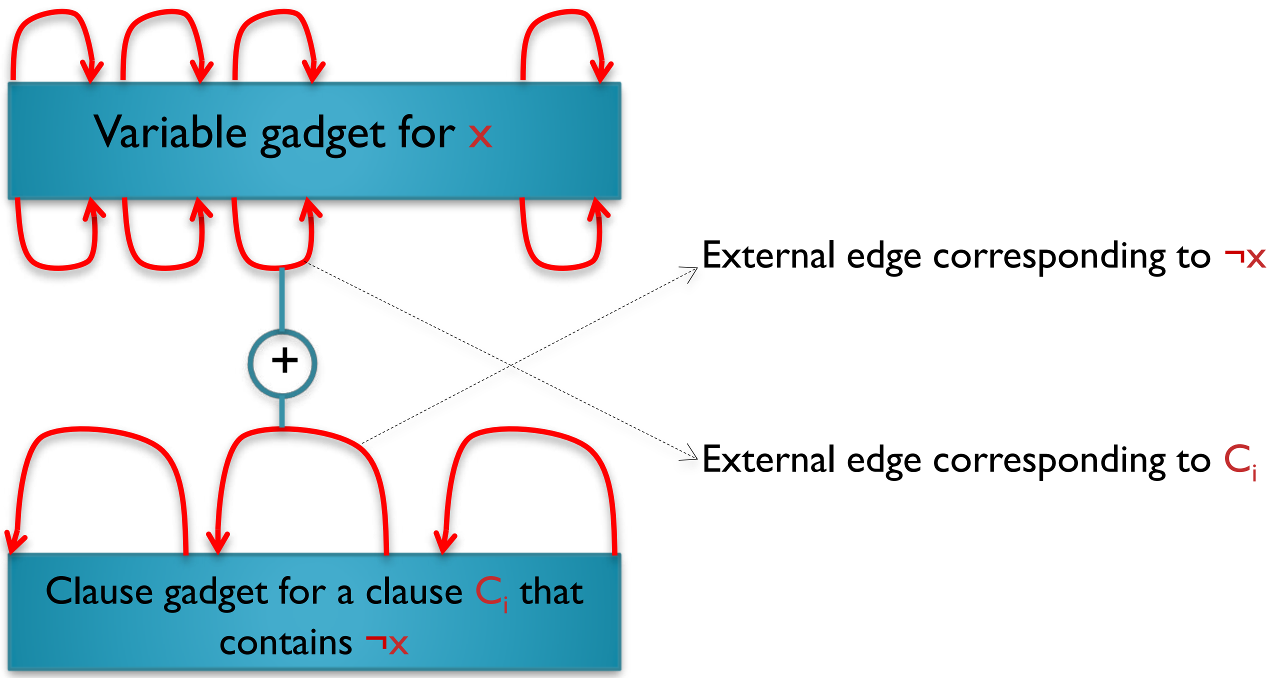
- Which of these 0/1 assignments to the variables correspond to actually contributing choice of the "touching patterns" of the XOR gadgets?

Construction of H



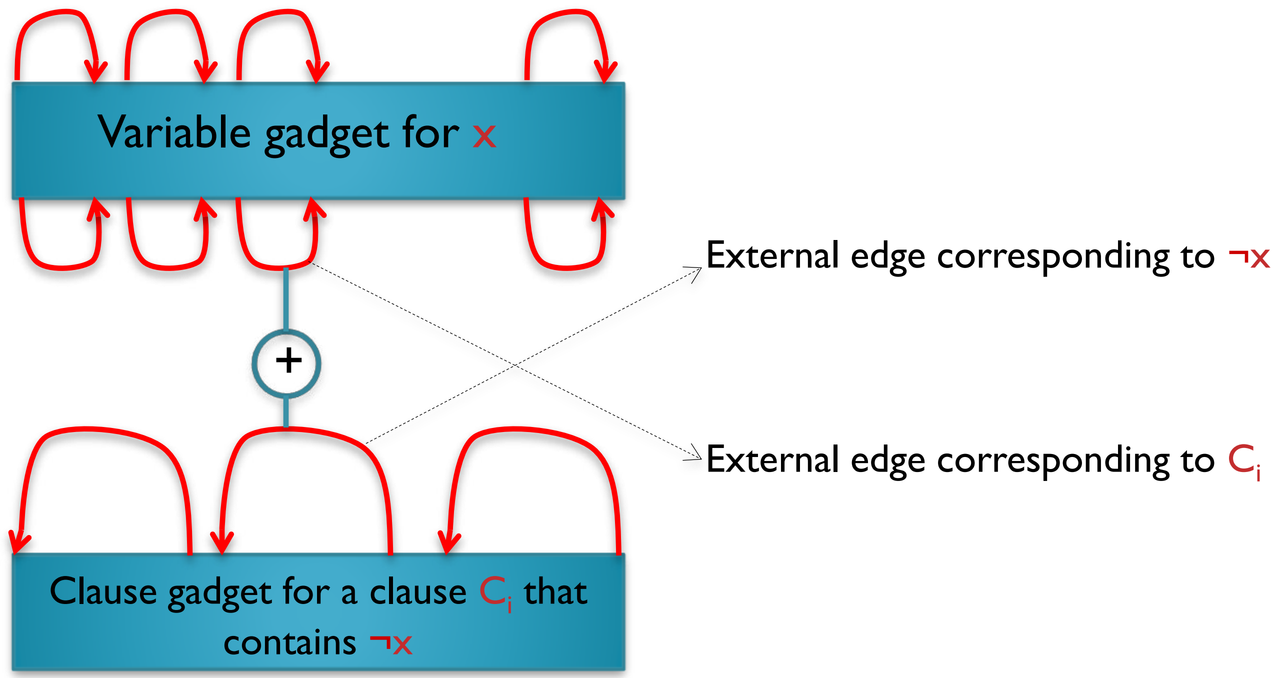
- Which of these 0/1 assignments to the variables correspond to <u>actually</u> contributing choice of the "touching patterns" of the XOR gadgets?
- Answer. Exactly the satisfying assignments of ϕ. *(Why?)*

Construction of H



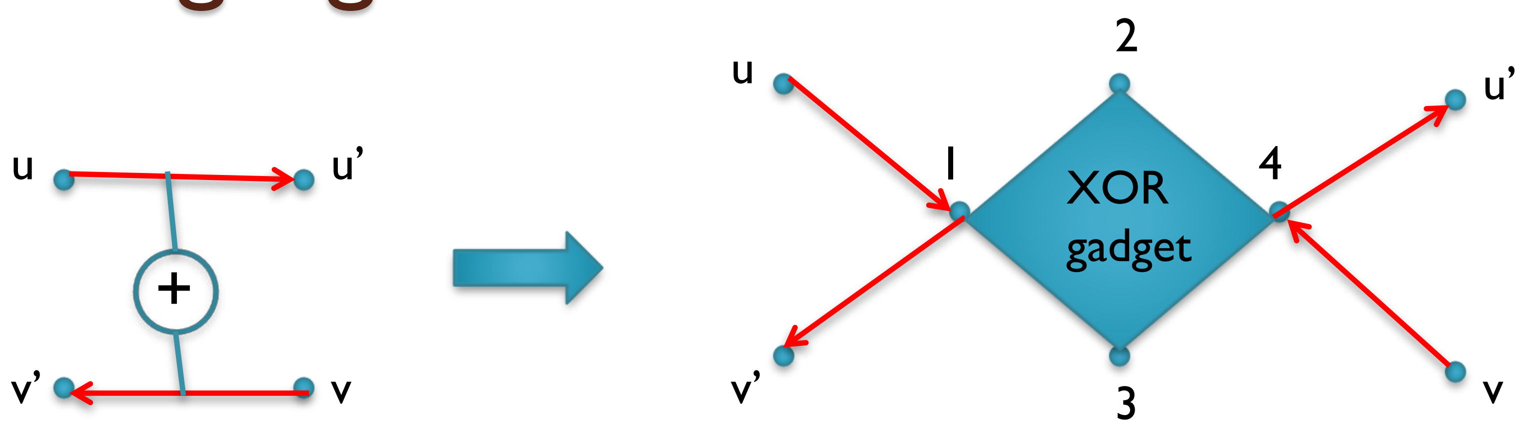
- Hence, the sum of the weighted cycle covers of H is 4^{3m}. #ϕ.
- In other words, $Perm(A_H) = 4^{3m}$. #ϕ. This concludes Step 1 of the proof of the Theorem.

Construction of H



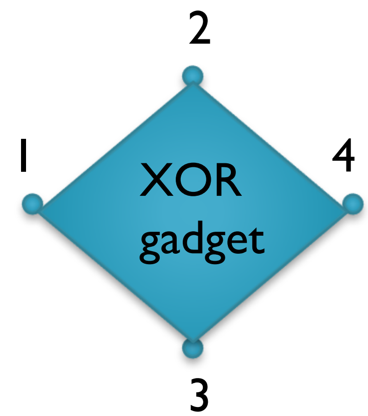
- Hence, the sum of the weighted cycle covers of H is 4^{3m}. #ϕ.
- In other words, $Perm(A_H) = 4^{3m}$. #ϕ. This conclues Step 1 of the proof of the Theorem. (Wait! How do we construct the XOR gadget?)

XOR gadget



- Let $X = (x_{i,j})_{4x4}$ be the adj. matrix of the XOR gadget.
- We need to pick $x_{i,j}$ in a way such that Feature 1, 2, 3 and 4 are satisfied.

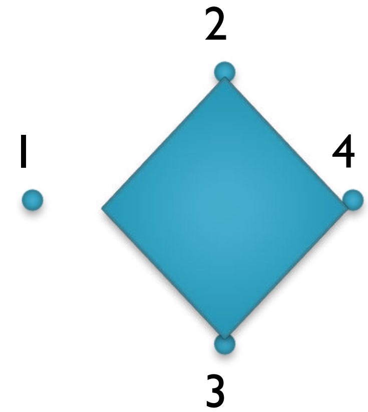
XOR gadget



- Let $X = (x_{i,j})_{4x4}$ be the adj. matrix of the XOR gadget.
- We need to pick $x_{i,j}$ in a way such that Feature 1, 2, 3 and 4 are satisfied.

- Condition 1. Feature 1 implies $Perm(X) = 0$.

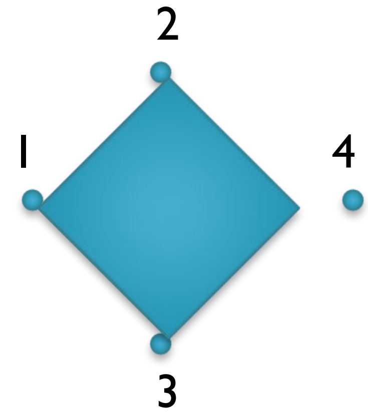
XOR gadget



- Let $X = (x_{i,j})_{4x4}$ be the adj. matrix of the XOR gadget.
- We need to pick $x_{i,j}$ in a way such that Feature 1, 2, 3 and 4 are satisfied.

- Condition 2. Feature 2 implies $Perm(X_{\{2,3,4\}}) = 0$, where $X_{\{2,3,4\}}$ is the submatrix of X restricted to the rows and columns that are indexed by 2, 3 and 4.

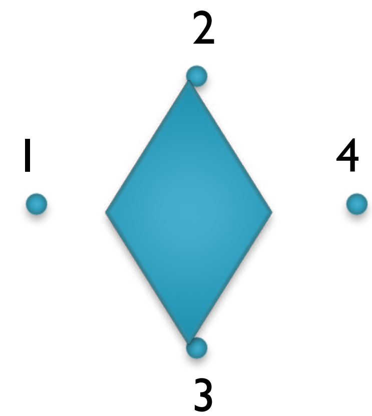
XOR gadget



- Let $X = (x_{i,j})_{4x4}$ be the adj. matrix of the XOR gadget.
- We need to pick $x_{i,j}$ in a way such that Feature 1, 2, 3 and 4 are satisfied.

- Condition 2. Feature 2 implies $Perm(X_{\{1,2,3\}}) = 0$, where $X_{\{1,2,3\}}$ is the submatrix of X restricted to the rows and columns that are indexed by 1, 2 and 3.

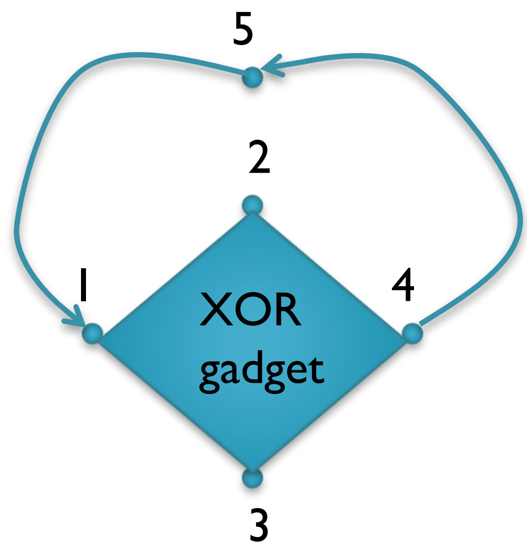
XOR gadget



- Let $X = (x_{i,j})_{4x4}$ be the adj. matrix of the XOR gadget.
- We need to pick $x_{i,j}$ in a way such that Feature 1, 2, 3 and 4 are satisfied.

- Condition 2. Feature 2 implies $Perm(X_{\{2,3\}}) = 0$, where $X_{\{2,3\}}$ is the submatrix of X restricted to the rows and columns that are indexed by 2 and 3.

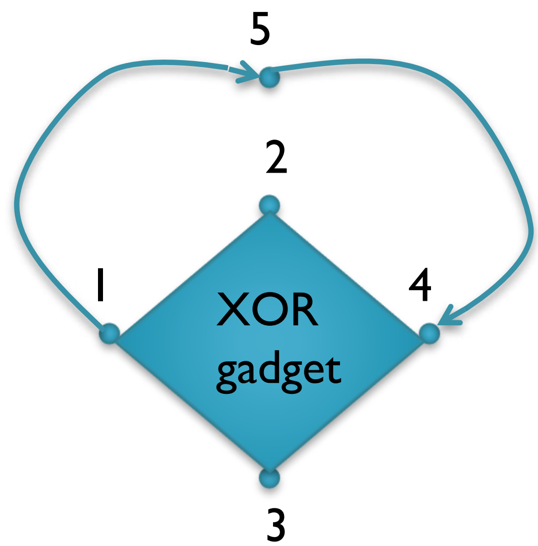
XOR gadget



- Let $X = (x_{i,j})_{4x4}$ be the adj. matrix of the XOR gadget.
- We need to pick $x_{i,j}$ in a way such that Feature 1, 2, 3 and 4 are satisfied.

- Condition 3. Feature 3 implies Perm(Y) = 4, where Y is the adjacency matrix of the above 5-vertex graph.

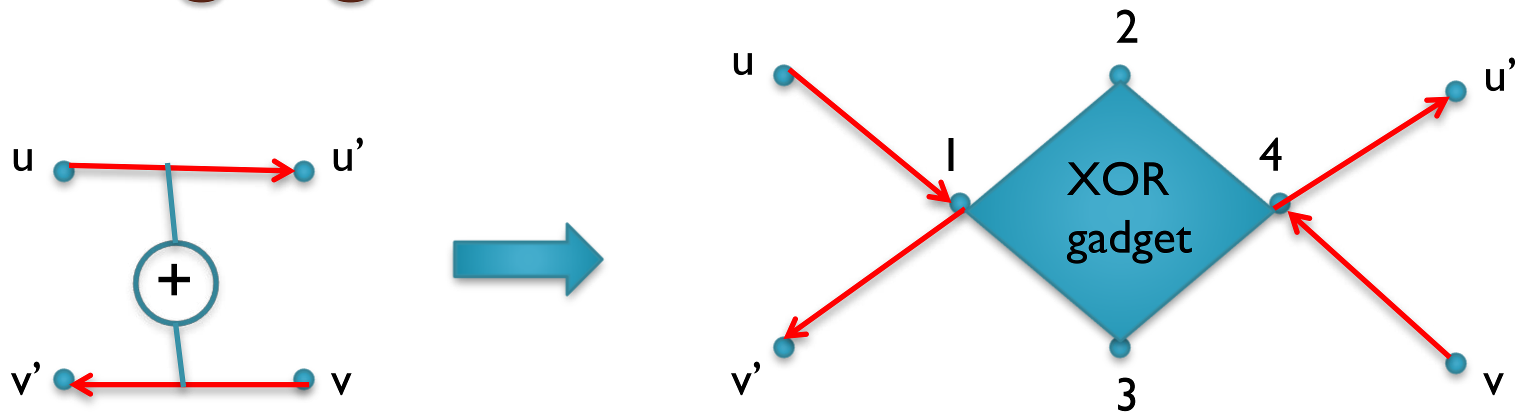
XOR gadget



- Let $X = (x_{i,j})_{4x4}$ be the adj. matrix of the XOR gadget.
- We need to pick $x_{i,j}$ in a way such that Feature 1, 2, 3 and 4 are satisfied.

- Condition 4. Feature 4 implies $Perm(Z) = 4$, where Z is the adjacency matrix of the above 5-vertex graph.

XOR gadget



- Set X as follows to satisfy Condition 1, 2, 3 and 4.

-

$X =$

0	1	-1	-1
1	-1	1	1
0	1	1	2
0	1	3	0

0/1-Permanent is #P-complete

- Theorem. *(Valiant 1979)* 0/1-Perm is #P-complete.
- Proof. Let ϕ be a 3CNF that has n variables and m clauses. Assume that every clause has <u>exactly</u> 3 literals.

- Step 1: From ϕ we'll form a graph $H = H_\phi$ that has edge weights in {-1, 0, 1, 2, 3} such that

$$\mathrm{Perm}(A_H) = \sum_{C:\ C \text{ is cycle cover of } H} \mathrm{wt}(C) = 4^{3m}.\#\phi \,.$$

- We have completed Step 1.

0/1-Permanent is #P-complete

- Theorem. *(Valiant 1979)* 0/1-Perm is #P-complete.
- Proof. Let ϕ be a 3CNF that has n variables and m clauses. Assume that every clause has <u>exactly</u> 3 literals.

- Step 2: We'll process H further to get a new graph $G = G_\phi$ with edge weights in $\{0,1\}$ such that $\#\phi$ can be efficiently computed from $\mathrm{Perm}(A_G)$.

- Let us now focus on Step 2.

Step 2

- Covert H to H' that has edge weights from {-1, 0, 1} by first introducing parallel edges, and then, introducing extra vertices to get rid of the parallel edges. Let p = poly(n,m) be the number of vertices of H'.

Step 2

- Covert H to H' that has edge weights from $\{-1, 0, 1\}$ by first introducing parallel edges, and then, introducing extra vertices to get rid of the parallel edges. Let $p = poly(n,m)$ be the number of vertices of H'.

- Observe that $Perm(A_H) = Perm(A_{H'}) \in [0, p!]$. Set $r = p^2$ and note that $2^r + 1 > p!$.

Step 2

- Covert H to H' that has edge weights from {-1, 0, 1} by first introducing parallel edges, and then, introducing extra vertices to get rid of the parallel edges. Let $p = poly(n,m)$ be the number of vertices of H'.

- Observe that $Perm(A_H) = Perm(A_{H'}) \in [0, p!]$. Set $r = p^2$ and note that $2^r + 1 > p!$.
- Hence, $Perm(A_{H'})$ is the same as $Perm(A_{H'}) \bmod (2^r+1)$.

Step 2

- Covert H to H' that has edge weights from $\{-1, 0, 1\}$ by first introducing parallel edges, and then, introducing extra vertices to get rid of the parallel edges. Let $p = poly(n,m)$ be the number of vertices of H'.

- Observe that $Perm(A_H) = Perm(A_{H'}) \in [0, p!]$. Set $r = p^2$ and note that $2^r + 1 > p!$.
- Hence, $Perm(A_{H'})$ is the same as $Perm(A_{H'}) \bmod (2^r+1)$.
- As $-1 = 2^r \bmod (2^r + 1)$, we can replace the weights of the edges in H' that are labelled by -1 with 2^r to form a graph G' and compute $Perm(A_{G'}) \bmod (2^r+1)$.

Step 2

- Covert H to H' that has edge weights from $\{-1, 0, 1\}$ by first introducing parallel edges, and then, introducing extra vertices to get rid of the parallel edges. Let $p = poly(n,m)$ be the number of vertices of H'.

- Finally, transform G' to G with $0/1$ edge weights by
  - replacing every edge with weight 2^r by a sequence of r edges each having weight 2, and then
  - replacing every edge with weight 2 by a pair of parallel weight 1 edges, and then
  - removing parallel edges like before.

Step 2

- Covert H to H' that has edge weights from {-1, 0, 1} by first introducing parallel edges, and then, introducing extra vertices to get rid of the parallel edges. Let p = poly(n,m) be the number of vertices of H'.

- In the end, we get $Perm(A_G) = 4^m . \#\phi \mod (2^r + 1)$, where G is a graph with 0/1 edge weights.

- It is because of the modulus "mod $(2^r + 1)$" that an FPRAS for 0/1-Perm doesn't imply an FPRAS for #3SAT.