Computational Complexity Theory

Lecture 8: Class EXP; Time Hierarchy; Ladner's theorem

Department of Computer Science,
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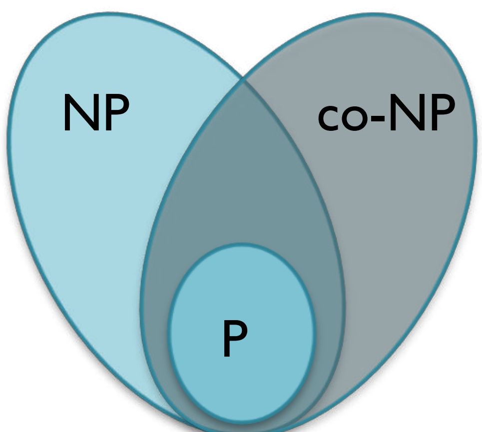
Class co-NP and EXP

Recap: Class co-NP

- Definition. For every $L \subseteq \{0,1\}^*$ let $\overline{L} = \{0,1\}^* \setminus L$.
  A language L is in co-NP if $\overline{L}$ is in NP.

- Example. $\overline{\text{SAT}} = \{\phi : \phi$ is *not* satisfiable$\}$.



Recap: Alternate definition of co-NP

- Recall, a language $L \subseteq \{0,1\}^*$ is in NP if there's a *poly-time verifier* M such that

$$x \in L \iff \exists u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x, u) = 1$$
$$x \in \bar{L} \iff \forall u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x, u) = 0$$
$$x \in \bar{L} \iff \forall u \in \{0,1\}^{p(|x|)} \text{ s.t. } \bar{M}(x, u) = 1$$

- Definition. A language $L \subseteq \{0,1\}^*$ is in co-NP if there's a polynomial function p and a *poly-time TM* M such that

$$x \in L \iff \forall u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x, u) = 1$$

for NP this was $\exists$

Recap: co-NP-completeness

- Definition. A language L' ⊆ {0,1}* is co-NP-complete if
  - L' is in co-NP
  - Every language L in co-NP is polynomial-time (Karp) reducible to L'.

- Theorem. $\overline{\text{SAT}}$ is co-NP-complete.

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  - ➢ L' is in co-NP
  - ➢ Every language L in co-NP is polynomial-time (Karp) reducible to L'.

- Theorem. Let

  TAUTOLOGY = {ϕ : every assignment satisfies ϕ }.

  TAUTOLOGY is co-NP-complete.

  Proof. Similar (homework)

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- Definition. A language $L' \subseteq \{0,1\}^*$ is co-NP-complete if
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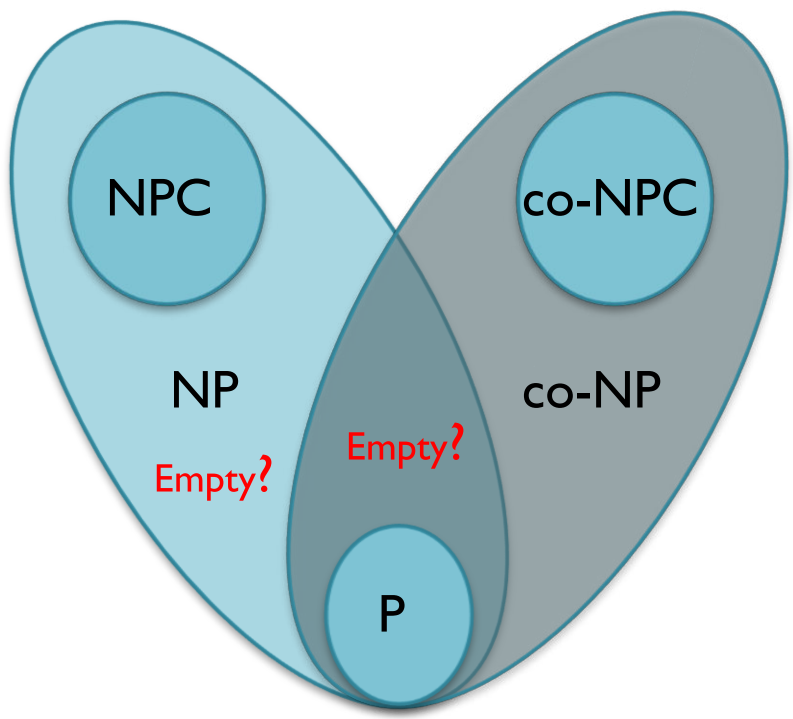
  - Every language L in co-NP is polynomial-time (Karp) reducible to L'.

- Theorem. If L in NP-complete then $\overline{L}$ is co-NP-complete

  Proof. Similar (homework)

Recap: The diagram again



If an NP-complete language belongs to co-NP then

$\Rightarrow$ NP $\subseteq$ co-NP
NP $=$ co-NP

Let C_1 and C_2 be two complexity classes.

If $C_1 \subseteq C_2$, then co-$C_1 \subseteq$ co-C_2.

Obs. co-(co-C) = C .

Recap: FACT in NP $\cap$ co-NP

- Integer factoring.

  FACT = {(N, U): there's a prime in [U] dividing N}

- Claim. FACT $\in$ NP $\cap$ co-NP

- So, FACT is NP-complete implies NP = co-NP.

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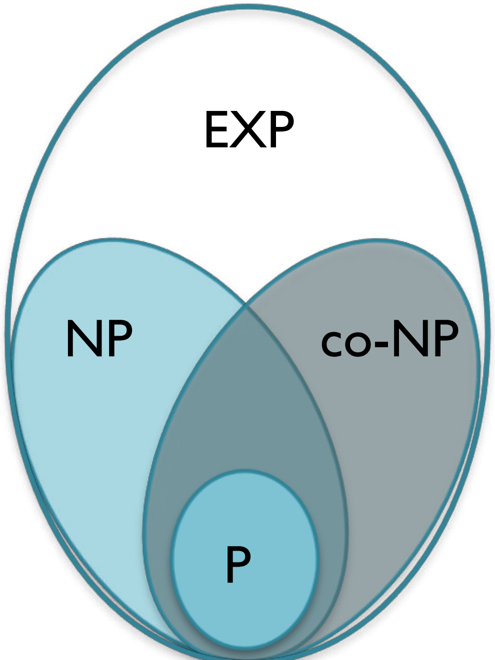

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- ***Exponential Time Hypothesis.*** *(Impagliazzo & Paturi 1999)* Any algorithm for 3-SAT takes $\geq 2^{\delta . n}$ time, where $\delta > 0$ is some fixed constant and n is the no. of variables.

In other words, δ cannot be made arbitrarily close to 0.

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ETH $\Rightarrow$ $P \neq NP$

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Homework: Read about Strong Exponential Time Hypothesis (SETH).

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We'll address this using ***diagonalization***

- **Observation.** $P \subseteq NP \subseteq EXP$

$$\boxed{\text{Is } P \subsetneq EXP\ ?}$$

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  1. There's a universal TM U that when given strings α and x, simulates M_α on x with only a <u>*small*</u> overhead.

     If M_α takes T time on x then U takes $O(T \log T)$ time to simulate M_α on x.

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- These techniques are characterized by <u>two</u> main features:
  1. There's a universal TM U that when given strings α and x, simulates M_α on x with only a <u>*small*</u> overhead.
  2. Every string represents some TM, and every TM can be represented by <u>*infinitely many*</u> strings.

Time Hierarchy Theorem

- An application of Diagonalization

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be <u>time-constructible</u> functions s.t.,

  $$f(n) \cdot \log f(n) = o(g(n)).$$

- Theorem. *(Hartmanis & Stearns 1965)*

  $$\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$$

- Theorem. $\mathrm{P} \subsetneq \mathrm{EXP}$

- This type of results are called **<u>lower bounds</u>**.

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  Task: Show that there's a language L decided by a TM D with time complexity $O(n^2)$ s.t., any TM M with runtime $O(n)$ cannot decide L.

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D's time steps not M_x's time steps.

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D outputs the **opposite** of what M_x outputs.

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  D runs in $O(n^2)$ time as n^2 is <u>time-constructible</u>.

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  Claim. There's no TM M with running time $O(n)$ that decides L (the language accepted by D).

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Contradiction! M does not decide L.

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- Theorem. $\mathrm{P} \subsetneq \mathrm{EXP}$

  Proof. Similar (homework)

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- Theorem. $\mathrm{P} \subsetneq \mathrm{EXP}$
- **No** EXP-complete problem (under poly-time Karp reduction) is in P.

E.g., Decide if a TM halts in k steps; generalized versions of games such as chess, checkers, Go, etc.

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- Conjecture. ***No*** algorithm solves 3SUM in $O(n^{2-\varepsilon})$ time for some constant $\varepsilon > 0$.

- However, there's a $\sim O(n^2 / (\log n)^2)$ time algorithm for 3SUM. ("~" suppressing a poly(log log n) factor.)

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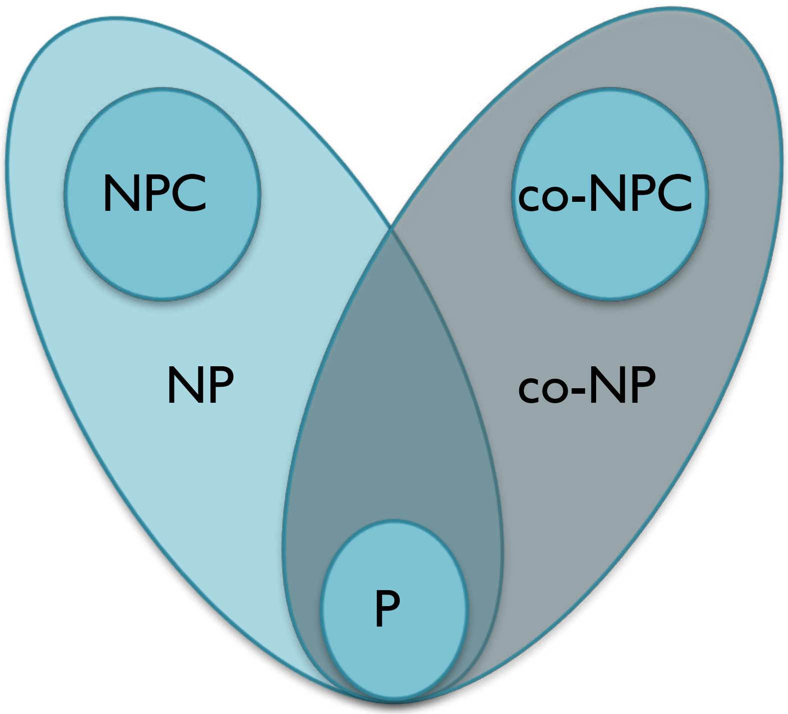
- Theorem *(Patrascu & Williams 2010)*. ETH implies kSUM requires $n^{\Omega(k)}$ time.

Revisiting NP∩co-NP



Conjecture: NP ≠ co-NP

⬇

P ≠ NP

General belief: P ≠ NP ∩ co-NP

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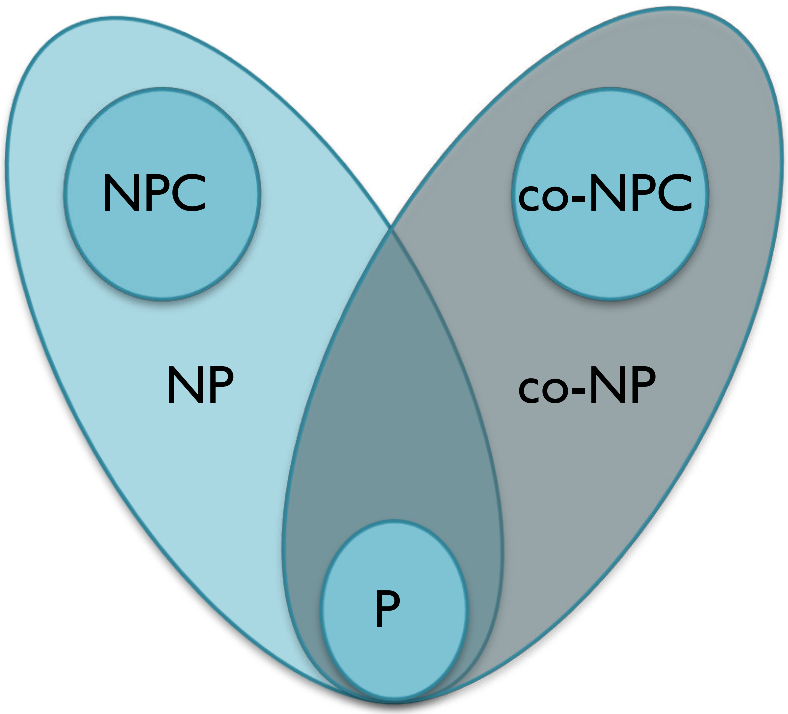


Conjecture: NP $\neq$ co-NP

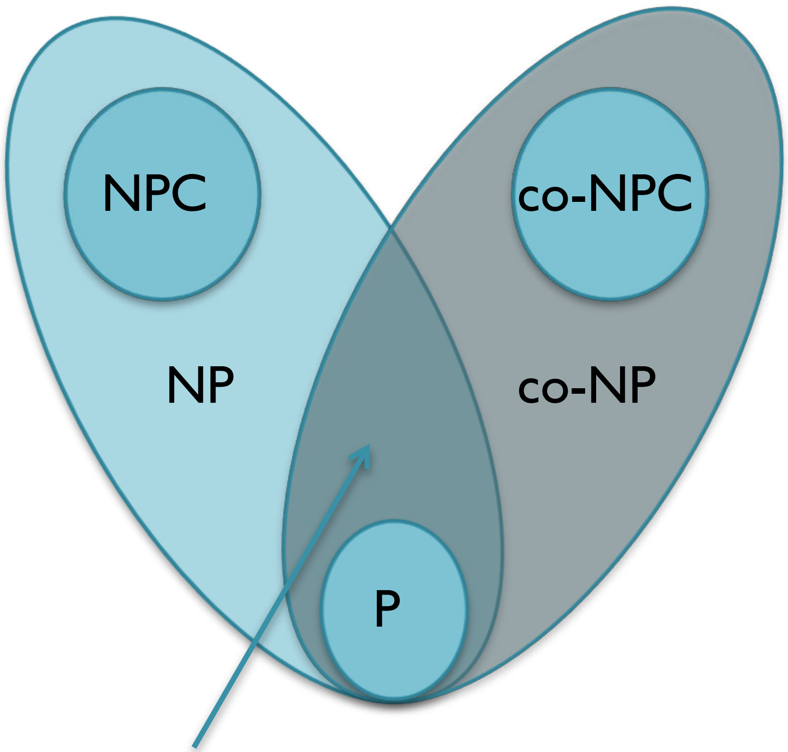
$\Downarrow$

P $\neq$ NP

General belief: $P \neq \underbrace{NP \cap \text{co-}NP}_{\text{Edmonds (1966)}}$

…conjectured $P = NP \cap \text{co-}NP$

Revisiting NP∩co-NP



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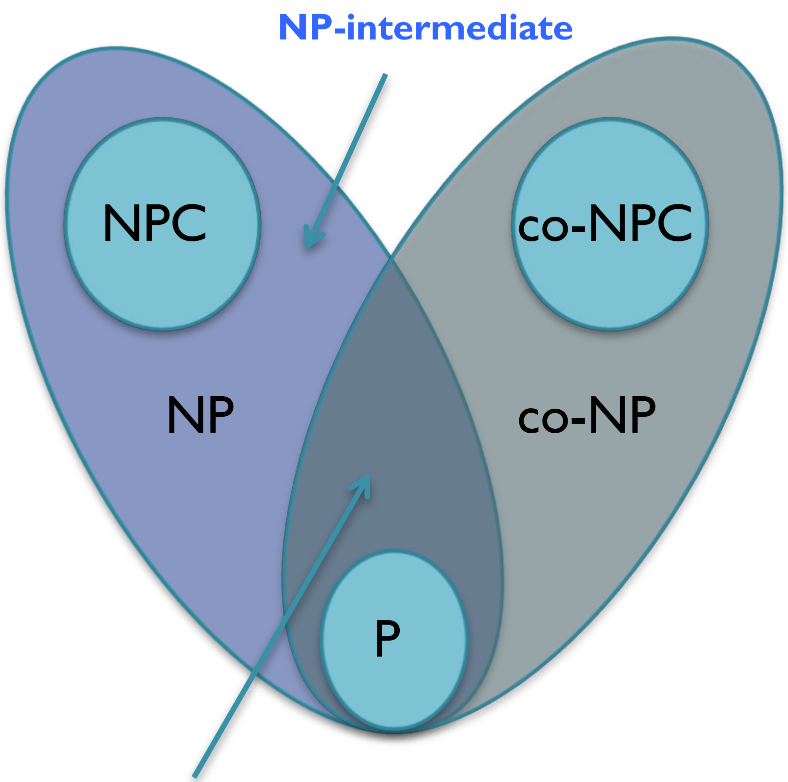
Check:
https://cstheory.stackexchange.com/questions/20021/reasons-to-believe-p-ne-np-cap-conp-or-not

Check if the shortest non-zero vector in an n-dimensional lattice has length at most 1 or at least $\sqrt{n}$.

- Integer factoring (FACT)
- Approximate shortest vector in a lattice

Ref: "*Lattice problems in NP∩co-NP*" by Aharonov & Regev (2005)

NP-intermediate problems



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Obs: If NP ≠ co-NP and FACT ∉ P then FACT is NP-intermediate.

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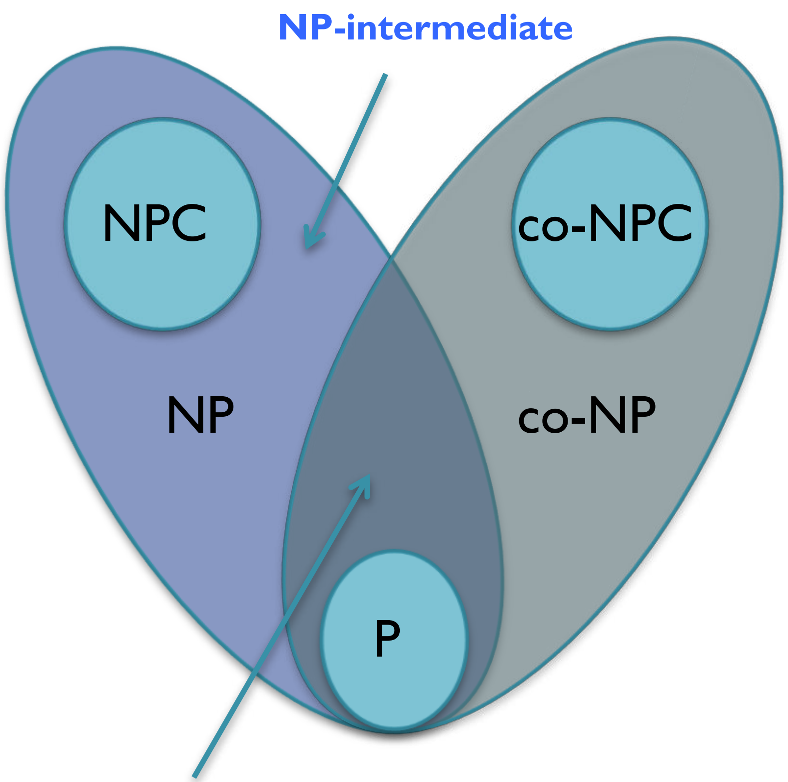

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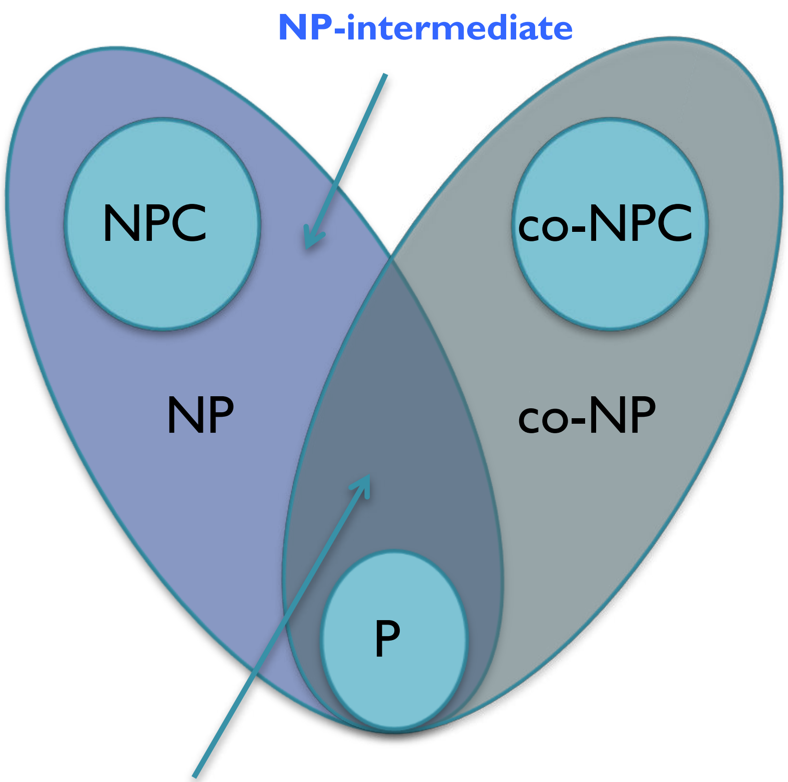
Ladner's theorem: P ≠ NP implies existence of a NP-intermediate language.

NP-intermediate problems



- Integer factoring (FACT)
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Conjecture: $NP \neq co\text{-}NP$

$\Downarrow$

$P \neq NP$

General belief: $P \neq NP \cap co\text{-}NP$

Obs: If $NP \neq co\text{-}NP$ and $FACT \notin P$ then FACT is NP-intermediate.

Ladner's theorem: $P \neq NP$ implies existence of a NP-intermediate language.

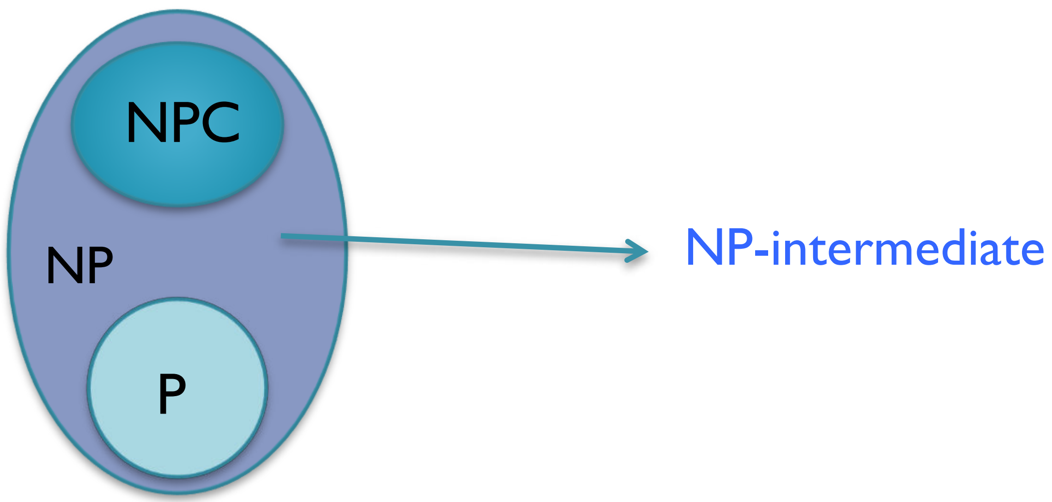
(proved using ***diagonalization***)

Ladner's Theorem

- Another application of Diagonalization

NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.



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  Proof. A delicate argument using diagonalization.

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H would be defined in such a way that SAT_H is NP-intermediate (assuming $P \neq NP$)

Ladner's theorem: Constructing H

- Theorem. There's a function $H: \mathbb{N} \rightarrow \mathbb{N}$ such that

    1. $H(m)$ is computable from m in $O(m^3)$ time.

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Proof: Later (uses diagonalization).

Let's see the proof of Ladner's theorem assuming the existence of such a "special" H.

Ladner's theorem: Proof

$$P \neq NP$$

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Ladner's theorem: Proof

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  - Compute $H(m)$, and construct the string $\phi\, 0\, 1^{m^{H(m)}}$
  - Check if $\phi\, 0\, 1^{m^{H(m)}}$ belongs to SAT_H.
- As $P \neq NP$, it must be that $SAT_H \notin P$.

Ladner's theorem: Proof

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- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \to \infty$ with m.
- This also implies a poly-time algorithm for SAT:

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Either $m \leq m_0$ (in which case the task reduces to checking if a constant-size Ψ is satisfiable),

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- On input ϕ, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.

or $H(m) > 2c$ (as $H(m)$ tends to infinity with m).

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- On input ϕ, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.
- Hence, w.l.o.g. $\quad |f(\phi)| \geq k > m^{2c}$

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- Hence, w.l.o.g. $n^c = |f(\phi)| \geq k > m^{2c}$

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> On input ϕ, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
> Compute $H(m)$ and check if $k = m^{H(m)}$.
> Hence, $\sqrt{n} \geq m$.

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- On input ϕ, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
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- Hence, $\sqrt{n} \geq m$. Also $\phi \in SAT$ iff $\Psi \in SAT$

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- Hence, $\sqrt{n} \geq m$. Also $\phi \in SAT$ iff $\Psi \in SAT$

Thus, checking if an n-size formula ϕ is satisfiable reduces to checking if a $\sqrt{n}$-size formula Ψ is satisfiable.

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Do this recursively! Only $O(\log \log n)$ recursive steps required.

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  - Compute $H(m)$ and check if $k = m^{H(m)}$.
  - Hence, $\sqrt{n} \geq m$. Also $\phi \in SAT$ iff $\Psi \in SAT$

- Hence SAT_H is not NP-complete, as $P \neq NP$.

Natural NP-intermediate problems ??

- Integer factoring
- Approximate shortest vector in a lattice
- Minimum Circuit Size Problem

  (“*Multi-output MCSP is NP-hard*”, Ilango, Loff & Oliveira 2020)
- Graph isomorphism

  (“*GI in QuasiP time*”, Babai 2015)