Computational Complexity Theory

Lecture 11: Polynomial Hierarchy (contd.)

Department of Computer Science,
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Recap: Problems between NP & PSPACE

- There are decision problems that don't appear to be captured by nondeterminism alone (i.e., with a **single** $\exists$ or $\forall$ quantifier), unlike problems in NP and co-NP.

- Example.

  Eq-DNF = {(φ,k): φ is a **DNF** and there's a DNF ψ of size $\leq$ k that is equivalent to φ}

- Is Eq-DNF in NP? …if we give a DNF ψ as a certificate, it is not clear how to efficiently verify that ψ and φ are equivalent. (W.l.o.g. $k \leq$ size of φ .)

Recap: Class Σ_2

- Definition. A language L is in Σ_2 if there's a polynomial function q(.) and a poly-time TM M (the "verifier") s.t.

$$x \in L \iff \exists u \in \{0,1\}^{q(|x|)} \ \forall v \in \{0,1\}^{q(|x|)} \text{ s.t. } M(x,u,v) = 1.$$

- Obs. Eq-DNF is in Σ_2.
- Proof. Think of u as another DNF ψ and v as an assignment to the variables. Poly-time TM M checks if ψ has size ≤ k and φ(v) = ψ(v).

- Remark. *(Masek 1979)* Even if φ is given by its truth-table, the problem (i.e., DNF-MCSP) is NP-complete.

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- Another example.

Succinct-SetCover = $\{(\varphi_1,\ldots\varphi_m,k)$: φ_i's are DNFs and there's an $S \subseteq [m]$ of size $\leq k$ s.t. $\bigvee_{i\in S} \varphi_i$ is a tautology$\}$

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- Obs. *(Homework)* Succinct-SetCover is in Σ_2.

- Other natural problems in PH: *"Completeness in the Polynomial-Time Hierarchy: A Compendium"* by *Schaefer and Umans (2008)*.

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- Obs. $P \subseteq NP \subseteq \Sigma_2$.

Recap: Class Σ_i

- Definition. A language L is in Σ_i if there's a polynomial function $q(.)$ and a poly-time TM M (the "verifier") s.t.

$$x \in L \Leftrightarrow \exists u_1 \in \{0,1\}^{q(|x|)} \; \forall u_2 \in \{0,1\}^{q(|x|)} \; Q_i u_i \in \{0,1\}^{q(|x|)}$$

$$\text{s.t. } M(x,u_1,\ldots,u_i) = 1,$$

where Q_i is $\exists$ or $\forall$ if i is odd or even, respectively.

- Obs. $\Sigma_i \subseteq \Sigma_{i+1}$ for every i.

Recap: Polynomial Hierarchy

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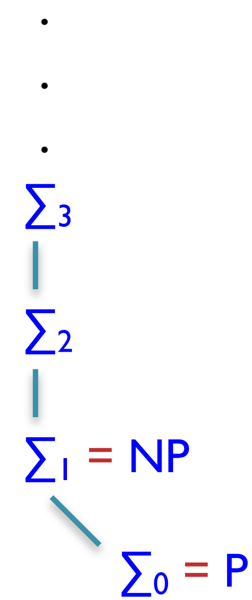
- Definition. *(Meyer & Stockmeyer 1972)*

$$PH = \bigcup_{i \in \mathbf{N}} \Sigma_i .$$



Recap: Class $\prod_i$

- Definition. $\prod_i = \text{co-}\sum_i = \{ L : \bar{L} \in \sum_i \}$.

- Obs. A language L is in $\prod_i$ if there's a polynomial function $q(.)$ and a poly-time TM M (the "verifier") s.t.

  $$x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)}\ \exists u_2 \in \{0,1\}^{q(|x|)} \ \ Q_i u_i \in \{0,1\}^{q(|x|)}$$

  $$\text{s.t. } M(x,u_1,\ldots, u_i) = 1,$$

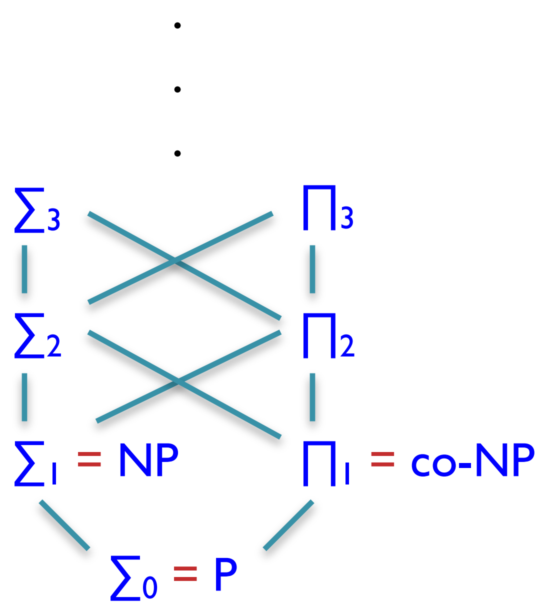
  where Q_i is $\forall$ or $\exists$ if i is odd or even, respectively.

- Obs. $\sum_i \subseteq \prod_{i+1} \subseteq \sum_{i+2}$.

Recap: Polynomial Hierarchy

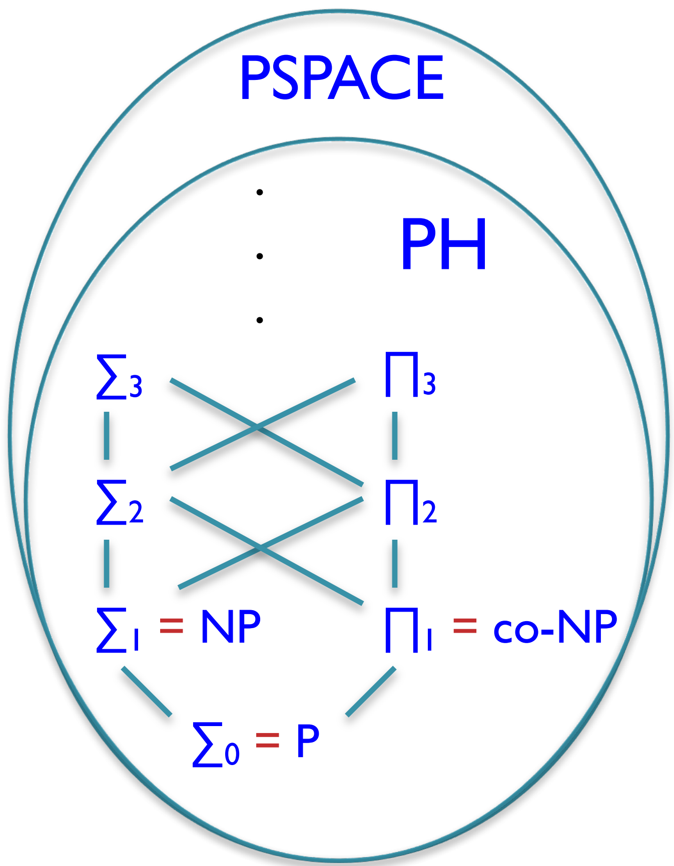
- Obs. $PH = \bigcup_{i \in \mathbf{N}} \Sigma_i = \bigcup_{i \in \mathbf{N}} \Pi_i$.

PH =



Recap: Polynomial Hierarchy

- Claim. PH ⊆ PSPACE .
- Proof. Similar to the proof of TQBF ∈ PSPACE.



Does PH collapse?

- General belief. Just as many of us believe $P \neq NP$ (i.e. $\Sigma_0 \neq \Sigma_1$) and $NP \neq co\text{-}NP$ (i.e. $\Sigma_1 \neq \Pi_1$), we also believe that for every i, $\Sigma_i \neq \Sigma_{i+1}$ and $\Sigma_i \neq \Pi_i$.
- Definition. We say PH collapses to the i-th level if $\Sigma_i = \Sigma_{i+1}$. (justified in the next theorem)
- Conjecture. There is no i such that PH collapses to the i-th level.

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- Conjecture. There is no i such that PH collapses to the i-th level.

This is stronger than the P $\neq$ NP conjecture.

PH collapse theorems

- Theorem. If $\Sigma_i = \Sigma_{i+1}$ then $PH = \Sigma_i$.

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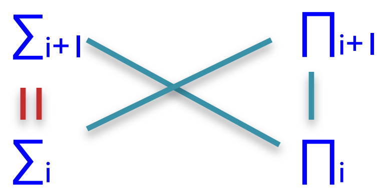

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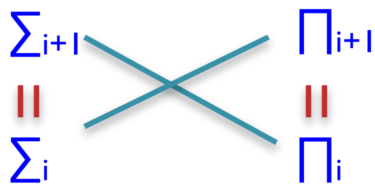

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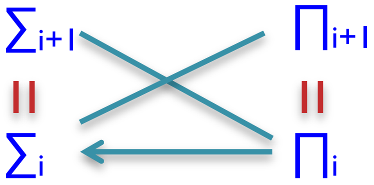

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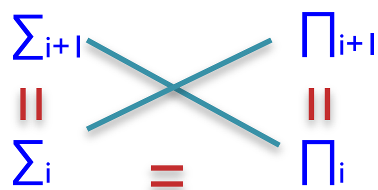

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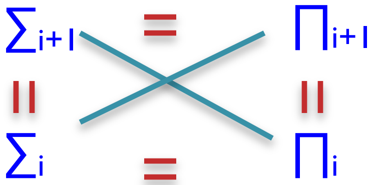

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  Goal is to show that $\Sigma_{i+1} = \Sigma_{i+2}$.

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- Clearly, L' is in $\Pi_{i+1} = \Sigma_i$.

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- Hence, L is a language in Σ_i .

Complete problems in PH ?

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is P = PH ? …use poly-time Karp reduction!

- Definition. A language L' is *PH-hard* if for every L in PH, $L \leq_p L'$. Further, if L' is in PH then L' is *PH-complete*.

Complete problems in PH ?

- Fact. If L is poly-time reducible to a language in Σ_i then L is in Σ_i .   (we've seen a similar fact for NP)

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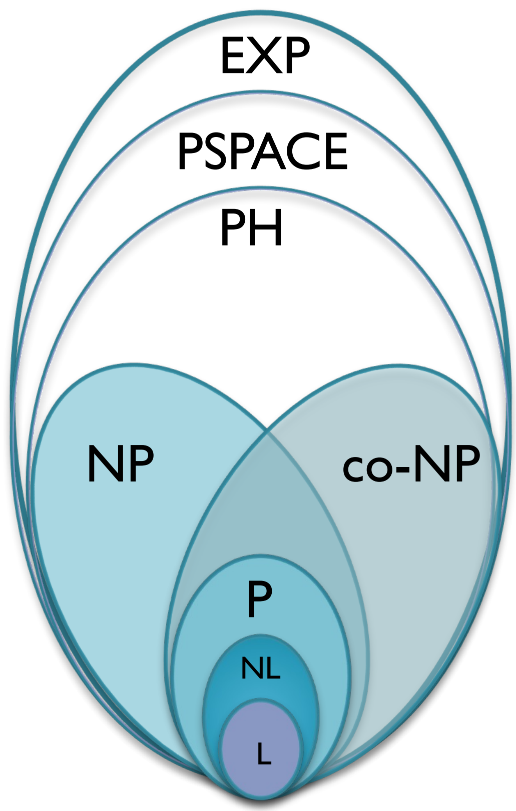
- Fact. If L is poly-time reducible to a language in Σ_i then L is in Σ_i .   (we've seen a similar fact for NP)
- Observation. If PH has a complete problem then PH collapses.
- Proof. If L is *PH-complete* then L is in Σ_i for some i. Now use the above fact to infer that PH = Σ_i .

Complete problems in PH ?

- Fact. If L is poly-time reducible to a language in Σ_i then L is in Σ_i . (we've seen a similar fact for NP)
- Corollary. PH $\subsetneq$ PSPACE unless PH collapses.



Complete problems in $\sum_i$

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- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is $P = \sum_i$? …use poly-time Karp reduction!

- Definition. A language L' is $\sum_i$ *-hard* if for every L in $\sum_i$, $L \leq_p L'$. Further, if L' is in $\sum_i$ then L' is $\sum_i$ *-complete*.

Complete problems in Σ_i

- Definition. The language Σ_i-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- Theorem. Σ_i-SAT is Σ_i *-complete*. (Σ_1-SAT is just SAT)

Complete problems in Σ_i

- Definition. The language Σ_i-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- Theorem. Σ_i-SAT is Σ_i *-complete*.
- Proof. Easy to see that Σ_i-SAT is in Σ_i .

$$x = \exists v_1 \forall v_2 \ldots Q_i v_i \; \varphi(v_1, \ldots, v_i) \in \Sigma_i\text{-SAT} \quad \Leftrightarrow$$

$$\exists u_1 \forall u_2 \ldots Q_i u_i \quad \text{s.t.} \quad M(x, u_1, \ldots, u_i) = 1,$$

where M outputs $\varphi(u_1, \ldots, u_i)$.

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- **Definition.** The language Σ_i-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- **Theorem.** Σ_i-SAT is Σ_i *-complete*.
- **Proof.** Let L be a language in Σ_i . Then there's a polynomial function $q(.)$ and a poly-time TM M s.t.

$$x \in L \Leftrightarrow \exists u_1 \forall u_2 \ldots Q_i u_i \quad \text{s.t.} \quad \underbrace{\varphi(x, u_1, \ldots, u_i)}_{\substack{\text{Boolean circuit} \\ \text{(by Cook-Levin)}}} = 1 .$$

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- Issue: φ needn't be a formula.

Complete problems in $\sum_i$

- Definition. The language $\sum_i$-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- Theorem. $\sum_i$-SAT is $\sum_i$ *-complete*.
- Proof. Let L be a language in $\sum_i$. Then there's a polynomial function $q(.)$ and a poly-time TM M s.t.

  $x \in L \Leftrightarrow \exists u_1 \forall u_2 \ldots Q_i u_i \ \ \varphi(x, u_1, \ldots, u_i)$ is true .

- Observation. From the proof of the Cook-Levin theorem, we can assume that φ is a CNF (if i is odd) or a DNF (if i is even). *(Homework)*

Complete problems in Σ_i

- Definition. The language Σ_i-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- Theorem. Σ_i-SAT is Σ_i *-complete*.
- Proof. Let L be a language in Σ_i . Then there's a polynomial function $q(.)$ and a poly-time TM M s.t.

$$x \in L \Leftrightarrow \exists u_1 \forall u_2 \ldots Q_i u_i \;\; \varphi(x, u_1, \ldots, u_i) \in \Sigma_i\text{-SAT} .$$

Other complete problems in Σ_2

- Ref. *"Completeness in the Polynomial-Time Hierarchy: A Compendium"* by *Schaefer and Umans (2008)*.

- Theorem. Eq-DNF and Succinct-SetCover are Σ_2 *-complete*.

An alternate characterization of PH

Oracle definition of Σ_i

- Definition. A language L is in $NP^{\Sigma_i\text{-SAT}}$ if there is a poly-time NTM with oracle access to Σ_i-SAT that decides L.

- Theorem. $\Sigma_{i+1} = NP^{\Sigma_i\text{-SAT}}$.

Oracle definition of Σ_i

- Definition. A language L is in $NP^{\Sigma_i\text{-SAT}}$ if there is a poly-time NTM with oracle access to Σ_i-SAT that decides L.

- Theorem. $\Sigma_{i+1} = NP^{\Sigma_i\text{-SAT}}$.

- Observe that Σ_1-SAT = SAT. We'll prove the special case $\Sigma_2 = NP^{SAT}$. The proof of the theorem is similar.

Oracle definition of Σ_i

- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in Σ_2. There's a polynomial function q(.) and a poly-time TM M s.t.

$x \in L \iff \exists u \in \{0,1\}^{q(|x|)}\ \forall v \in \{0,1\}^{q(|x|)}$ s.t. $M(x,u,v) = 1$.

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$$x \in L \iff \exists u \in \{0,1\}^{q(|x|)} \ \forall v \in \{0,1\}^{q(|x|)} \text{ s.t. } \underbrace{\varphi(x,u,v)}_{\text{Boolean circuit}} = 1.$$

Boolean circuit
(*by Cook-Levin*)

- In fact, owing to the proof of the Cook-Levin theorem, we can assume that φ is a DNF.

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  $x \in L \Leftrightarrow \exists u \in \{0,1\}^{q(|x|)}$ $\forall v \in \{0,1\}^{q(|x|)}$ s.t. $\neg\varphi(x,u,v) = 0$.

- Think of a NTM N that has the knowledge of M. On input x, it guesses $u \in \{0,1\}^{q(|x|)}$ non-deterministically and computes the circuit $\varphi(x,u,v)$. Then, it queries the SAT oracle with $\neg\varphi(x,u,v)$.

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- Note that $\neg\varphi(x,u,v)$ is a CNF.

Oracle definition of Σ_i

- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
- Special case: N asks at most **one** query to the SAT oracle on every computation path on input x.

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- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
- Special case: N asks at most **one** query to the SAT oracle on every computation path on input x.
- We need to construct a Σ_2-statement that captures N's computation on input x.

Oracle definition of $\sum_i$

- Theorem. $\sum_2 = NP^{SAT}$.
- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
- Special case: N asks at most one query to the SAT oracle on every computation path on input x.
- Think of a TM M that takes input x and $w \in \{0,1\}^{q(|x|)}$, $a_1 \in \{0,1\}$ and $u_1, v_1 \in \{0,1\}^{q(|x|)}$, where q(|x|) is the runtime of N on input x, and does the following:

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- M simulates N on input x with w as the non-deterministic choices.

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- Think of a TM M that takes input x and $w \in \{0,1\}^{q(|x|)}$, $a_1 \in \{0,1\}$ and u_1, $v_1 \in \{0,1\}^{q(|x|)}$, where $q(|x|)$ is the runtime of N on input x, and does the following:
- M simulates N on input x with w as the <u>computation path</u>. Suppose φ is the query asked by N on the path of computation defined by w.

Oracle definition of Σ_i

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  - If $a_1 = 1$ and $\varphi(u_1) = 1$, M continues the simulation; else it stops and outputs 0. (In this case, M ignores v_1.)

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- Think of a TM M that takes input x and $w \in \{0,1\}^{q(|x|)}$, $a_1 \in \{0,1\}$ and $u_1, v_1 \in \{0,1\}^{q(|x|)}$, where $q(|x|)$ is the runtime of N on input x, and does the following:
- At the end of the simulation, M outputs whatever N outputs. Note: M is a poly-time TM.

Oracle definition of Σ_i

- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
- Special case: N asks at most one query to the SAT oracle on every computation path on input x.
- Observation. For any $w \in \{0,1\}^{q(|x|)}$ and $a_1 \in \{0,1\}$,

➢ N on computation path w gets answer a_1 from the SAT oracle and accepts x ⟺

$\exists u_1 \in \{0,1\}^{q(|x|)}$ $\forall v_1 \in \{0,1\}^{q(|x|)}$ s.t. $M(x,w,a_1,u_1,v_1) = 1$.

(…will prove the observation shortly. Let's finish the proof.)

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- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
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- $x \in L \iff \exists w \in \{0,1\}^{q(|x|)}, a_1 \in \{0,1\}$ s.t

➢ N on computation path w gets answer a_1 from the SAT oracle and accepts x $\iff \exists w \in \{0,1\}^{q(|x|)}, a_1 \in \{0,1\}$ $\exists u_1 \in \{0,1\}^{q(|x|)}\ \forall v_1 \in \{0,1\}^{q(|x|)}$ s.t. $M(x,w,a_1,u_1,v_1) = 1$.

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➢ N on computation path w gets answer a_1 from the SAT oracle and accepts x $\Leftrightarrow \exists w \in \{0,1\}^{q(|x|)}, a_1 \in \{0,1\}$

$\exists u_1 \in \{0,1\}^{q(|x|)}\ \forall v_1 \in \{0,1\}^{q(|x|)}$ s.t. $M(x, w, \underbrace{a_1, u_1}_{\text{Call it } u}, v_1) = 1$.

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- Special case: N asks at most one query to the SAT oracle on every computation path on input x.
- $x \in L \iff \exists w \in \{0,1\}^{q(|x|)}, a_1 \in \{0,1\}$ s.t

➢ N on computation path w gets answer a_1 from the SAT oracle and accepts x $\iff$

$\exists u \in \{0,1\}^{2q(|x|)+1}\ \forall v_1 \in \{0,1\}^{q(|x|)}$ s.t. $M(x,u,v_1) = 1$.

- Therefore, L is in Σ_2 .

Proof of the observation

- Observation. For any $w \in \{0,1\}^{q(|x|)}$ and $a_1 \in \{0,1\}$,
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- Proof.(⇒) M simulates N on computation path w. Let φ be the query asked by N to SAT.
- If $a_1 = 1$, $\exists u_1 \in \{0,1\}^{q(|x|)}$ $\varphi(u_1) = 1$ and N accepts x.

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- If $a_1 = 0$, $\forall v_1 \in \{0,1\}^{q(|x|)}$ $\varphi(v_1) = 0$ and N accepts x.

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- Proof.(⟹) M simulates N on computation path w. Let φ be the query asked by N to SAT.
- Irrespective of the value of a_1,

$\exists u_1 \in \{0,1\}^{q(|x|)}$ $\forall v_1 \in \{0,1\}^{q(|x|)}$ s.t. $M(x,w,a_1,u_1,v_1) = 1$.

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- Proof.(⟸) Need to show that N on computation path w gets answer a_1 from the SAT oracle. (*Homework*)

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- Theorem. $\Sigma_2 = NP^{SAT}$.
- Proof. Let L be a language in NP^{SAT}. There's a NTM N that decides L with oracle access to SAT.
- General case: N asks at most $q(|x|)$ queries to SAT oracle on every computation path on input x.
- *Homework*: Prove the general case. Define the poly-time machine M appropriately.

Oracles versus efficient algorithms

- Definition. A language L is in P^{SAT} if there is a poly-time TM with oracle access to SAT that decides L.
- $\Delta_2 := P^{SAT} \subseteq \Sigma_2 \cap \Pi_2$.
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- Yet, in the first case we believe $P^{SAT} \neq NP^{SAT}$, whereas in the second case PH collapses to P, i.e., $P^{SAT} = NP^{SAT}$.
- Why? Think to understand the difference between oracles and poly-time algorithms for SAT (*Homework*).