Computational Complexity Theory

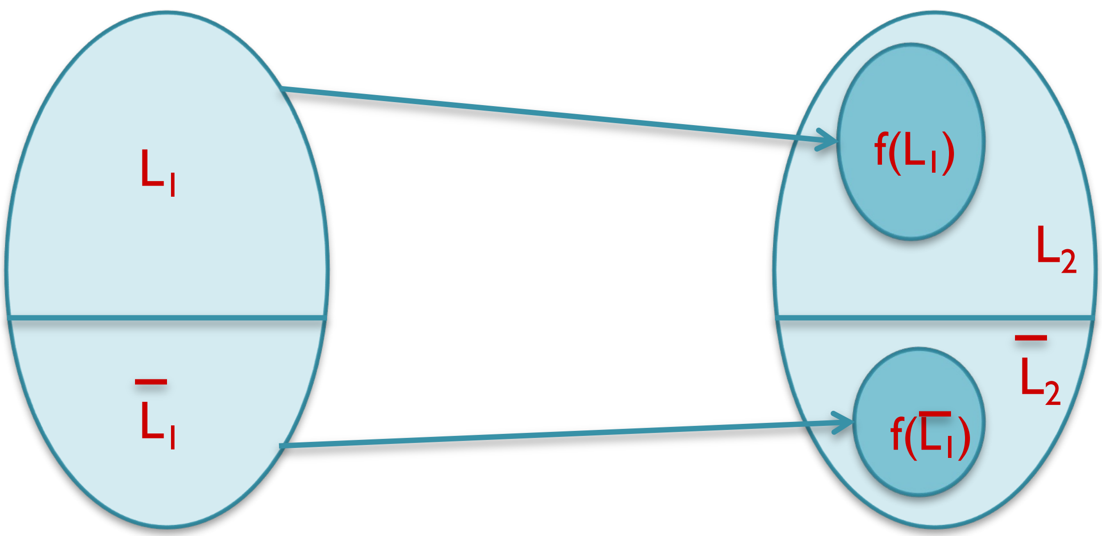
Lecture 4: More NP-complete problems; Decision versus Search

Department of Computer Science,
Indian Institute of Science

Recap: Polynomial-time reduction

- Definition. We say a language $L_1 \subseteq \{0,1\}^*$ is *polynomial-time (Karp) reducible* to a language $L_2 \subseteq \{0,1\}^*$ if there's a polynomial-time computable function f s.t.

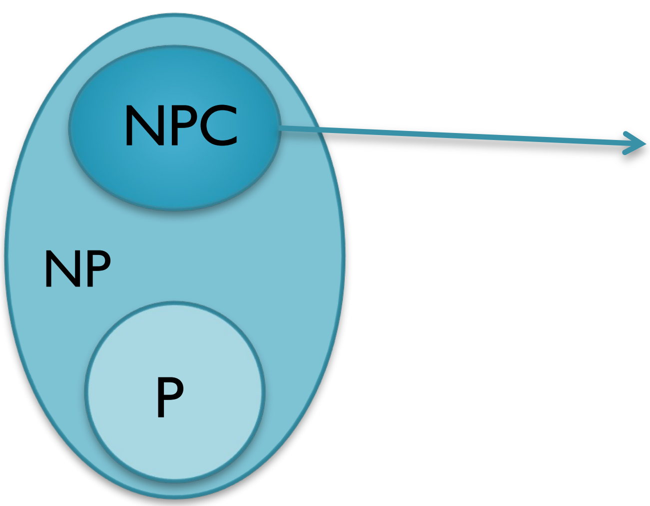
$$x \in L_1 \Leftrightarrow f(x) \in L_2$$



Recap: NP-completeness

- **Definition.** A language L' is *NP-hard* if for every L in NP, $L \leq_p L'$. Further, L' is *NP-complete* if L' is in NP and is NP-hard.

- **Observe.** If L' is NP-hard and L' is in P then P = NP. If L' is NP-complete then L' in P if and only if P = NP.



Hardest problems inside NP in the sense that if one NPC problem is in P then all problems in NP is in P.

Recap: Few words on reductions

- As to how we define a reduction from one language to the other (or one function to the other) is usually guided by a *<u>question on</u> whether two <u>complexity classes</u> are different or identical*.

- For polynomial-time reductions, the question is whether or not P equals NP.

- Reductions help us define *complete problems* (the 'hardest' problems in a class) which in turn help us compare the complexity classes under consideration.

Class NP : Examples

- Vertex cover (NP-complete)
- 0/1 integer programming (NP-complete)
- 3-coloring planar graphs (NP-complete)
- 2-Diophantine solvability (NP-complete)
- Integer factoring (unlikely to be NP-complete)
- Graph isomorphism (Quasi-P)

Recap: Existence of an NPC problem

- Let L' = { $(\alpha, x, 1^m, 1^t)$: there exists a $u \in \{0,1\}^m$ s.t. M_α accepts (x, u) in t steps }

- Observation. L' is NP-complete.

- The language L' involves Turing machine in its definition. Next, we'll see an example of an NP-complete problem that is arguably more natural.

Recap: A natural NP-complete problem

- Definition. A Boolean formula is in *Conjunctive Normal Form* (*CNF*) if it is an AND of OR of literals.

  e.g. $\varphi = (x_1 \vee x_2) \wedge (x_3 \vee \neg x_2)$

- Definition. Let SAT be the language consisting of all *satisfiable CNF formulae*.

- Theorem. *(Cook 1971, Levin 1973)* SAT is NP-complete.

  Easy to see that SAT is in NP.

  Need to show that SAT is NP-hard.

Recap: Cook-Levin theorem

- Main idea: Computation is ***local***; i.e., every step of computation *looks at* and *changes* only constantly many bits; and this step can be implemented by a small CNF formula.

- Let $L \in NP$. We intend to come up with a polynomial-time computable function $f: x \longmapsto \varphi_x$ s.t.,
  - $x \in L \iff \varphi_x \in SAT$
    - Notation: $|\varphi_x|$:= size of φ_x

      = number of $\vee$ or $\wedge$ in φ_x

Recap: Cook-Levin theorem

- Language L has a poly-time verifier M such that

$$x \in L \iff \exists u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x, u) = 1$$

- Idea: For any fixed x, we can capture the computation of M(x, ..) by a CNF φ_x such that

$$\exists u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x, u) = 1 \iff \varphi_x \text{ is satisfiable}$$

- For any fixed x, M(x, ..) is a deterministic TM that takes u as input and runs in time polynomial in $|u|$.

Recap: Cook-Levin theorem

- Main Theorem. Let N be a deterministic TM that runs in time T(n) on every input u of length n, and outputs 0/1. Then,
  1. There's a CNF φ(u, "*auxiliary variables*") of size *poly*(T(n)) such that for every u, φ(u, "*auxiliary variables*") is satisfiable as a function of the "*auxiliary variables*" if and only if N(u) = 1.
  2. φ is computable in time *poly(T(n))* from N, T & n.

- φ(u, "*auxiliary variables*") is satisfiable as a function of **all** the variables if and only if ∃u s.t N(u) = 1.

Recap: Main theorem

- Step 1. Let N be a deterministic TM that runs in time T(n) on every input u of length n, and outputs 0/1. Then,
  1. There's a Boolean circuit ψ of size *poly*(T(n)) such that ψ(u) = 1 if and only if N(u) =1.
  2. ψ is computable in time *poly(T(n))* from N, T & n.

- Step 2. "Convert" circuit ψ to a CNF φ efficiently by introducing <u>auxiliary variables</u>.

NP complete problems: Examples

- Independent Set
- Clique
- Vertex cover
- 0/1 integer programming
- Max-Cut (NP-hard)

Karp 1972

- 3-coloring planar graphs *Stockmeyer 1973*
- 2-Diophantine solvability *Adleman & Manders 1975*

Ref: Garey & Johnson, "*Computers and Intractability*" 1979

NPC problems from number theory

- SqRootMod: Given natural numbers a, b and c, check if there exists a natural number $x \leq c$ such that

$$x^2 = a \pmod{b} .$$

- Theorem: SqRootMod is NP-complete.

Manders & Adleman 1976

NPC problems from number theory

- Variant_IntFact : Given natural numbers L, U and N, check if there exists a **natural number** $d \in [L, U]$ such that d divides N.
- Claim: Variant_IntFact is NP-hard under *randomized poly-time reduction*.
- Reference: *https://cstheory.stackexchange.com/questions/4769/an-np-complete-variant-of-factoring/4785*

A peculiar NP problem

- Minimum Circuit Size Problem (MCSP): Given the **truth table** of a Boolean function f and an integer s, check if there is a circuit of size $\leq$ s that computes f.

- Easy to see that MCSP is in NP.

- Is MCSP NP-complete? Not known!

A peculiar NP problem

- Minimum Circuit Size Problem (MCSP): Given the **truth table** of a Boolean function f and an integer s, check if there is a circuit of size ≤ s that computes f.

- Easy to see that MCSP is in NP.

- Is MCSP NP-complete? Not known!
- Multi-output MCSP is NP-hard under poly-time randomized reductions. *(Ilango, Loff, Oliveira 2020)*

A peculiar NP problem

- Minimum Circuit Size Problem (MCSP): Given the **truth table** of a Boolean function f and an integer s, check if there is a circuit of size ≤ s that computes f.

- Easy to see that MCSP is in NP.

- Is MCSP NP-complete? Not known!
- Partial fn. MCSP is NP-hard under poly-time randomized reductions. *(Hirahara 2022)*

More NP-complete problems

Example 1: Independent Set

- INDSET := {(G, k): G has independent set of size k}

- Goal: Design a poly-time reduction f s.t.

$$x \in 3SAT \iff f(x) \in INDSET$$

- Reduction from 3SAT: Recall, a reduction is just an efficient algorithm that takes input a 3CNF φ and outputs a (G, k) tuple s.t

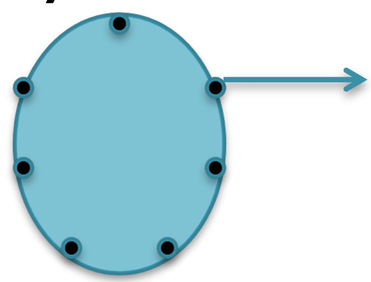
$$\varphi \in 3SAT \iff (G, k) \in INDSET$$

Example 1: Independent Set

- Reduction: Let φ be a 3CNF with m clauses and n variables. Assume, every clause has exactly 3 literals.

Example 1: Independent Set

- Reduction: Let φ be a 3CNF with m clauses and n variables. Assume, every clause has exactly 3 literals.

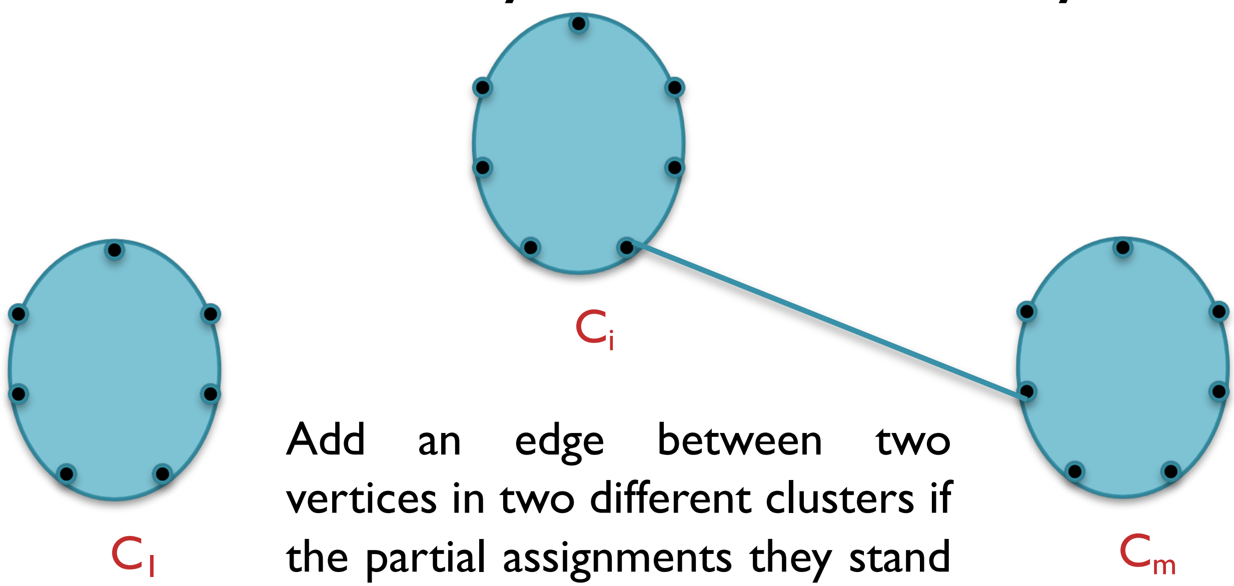


A vertex stands for a partial assignment of the variables in C_i that satisfies the clause

For every clause C_i form a complete graph (cluster) on 7 vertices

Example 1: Independent Set

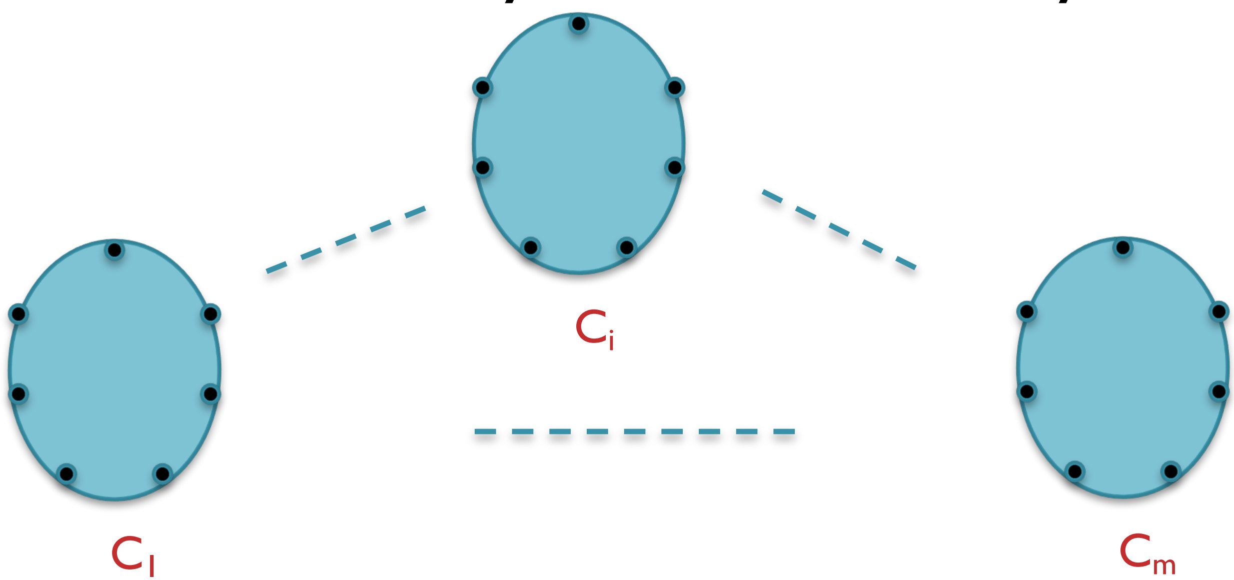
- Reduction: Let φ be a 3CNF with m clauses and n variables. Assume, every clause has exactly 3 literals.



Add an edge between two vertices in two different clusters if the partial assignments they stand for are <u>incompatible</u>.

Example 1: Independent Set

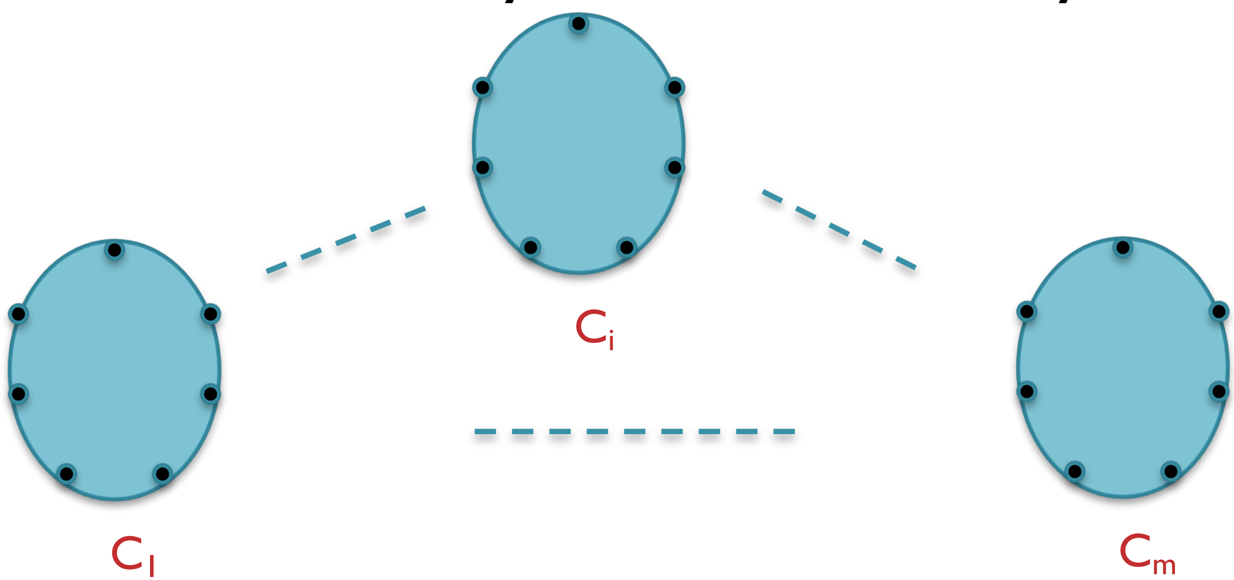
- Reduction: Let φ be a 3CNF with m clauses and n variables. Assume, every clause has exactly 3 literals.



Graph G on 7m vertices

Example 1: Independent Set

- Reduction: Let φ be a 3CNF with m clauses and n variables. Assume, every clause has exactly 3 literals.



- Obs: φ is satisfiable iff G has an ind. set of size m.

Example 2: Clique

- CLIQUE := {(H, k): H has a clique of size k}

- Goal: Design a poly-time reduction f s.t.

$$x \in \text{INDSET} \iff f(x) \in \text{CLIQUE}$$

- Reduction from INDSET: The reduction algorithm computes $\overline{G}$ from G

$$(G, k) \in \text{INDSET} \iff (\overline{G}, k) \in \text{CLIQUE}$$

Example 3: Vertex Cover

- VCover := {(H, k): H has a vertex cover of size k}

- Goal: Design a poly-time reduction f s.t.

$$x \in \text{INDSET} \iff f(x) \in \text{VCover}$$

- Reduction from INDSET: Let n be the number of vertices in G. The reduction algorithm maps (G, k) to (G, n-k).

$$(G, k) \in \text{INDSET} \iff (G, n-k) \in \text{VCover}$$

Example 4: 0/1 Integer Programming

- 0/1 IProg := Set of satisfiable 0/1 integer programs
- A 0/1 integer program is a set of linear inequalities with rational coefficients and the variables are allowed to take only 0/1 values.

- Reduction from 3SAT: A clause is mapped to a linear inequality as follows

$$x_1 \lor \overline{x}_2 \lor x_3 \longrightarrow x_1 + (1 - x_2) + x_3 \geq 1$$

Example 5: Max Cut

- MaxCut : Given a graph find a <u>cut</u> with the max size.
- A <u>*cut*</u> of $G = (V, E)$ is a tuple $(U, V \setminus U)$, $U \subseteq V$. <u>Size</u> of a cut $(U, V \setminus U)$ is the number of edges from U to $V \setminus U$.

- MinVCover: Given a graph H, find a vertex cover in H that has the min size.

- Obs: From MinVCover(H), we can readily check if $(H, k) \in$ VCover, for any k.

Example 5: Max Cut

- MaxCut : Given a graph find a cut with the max size.
- A *cut* of $G = (V, E)$ is a tuple $(U, V \setminus U)$, $U \subseteq V$. Size of a cut $(U, V \setminus U)$ is the number of edges from U to $V \setminus U$.

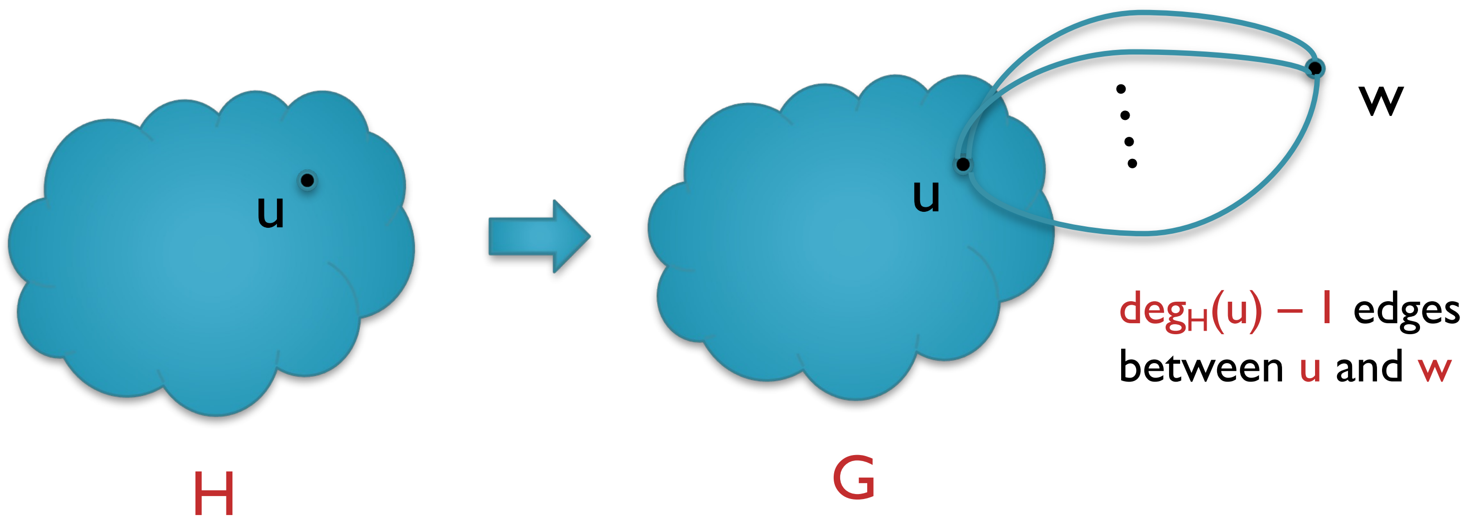
- Goal: A poly-time reduction from MinVCover to MaxCut.

$$H \xrightarrow{f} G \quad \text{s.t.}$$

$$\boxed{\text{Size of a MaxCut}(G) = 2.|E(H)| - |\text{MinVCover}(H)|}$$

Example 5: Max Cut

- The reduction: H $\xrightarrow{f}$ G



- G is formed by adding a new vertex w and adding $deg_H(u) - 1$ edges between every $u \in V(H)$ and w.

Example 5: Max Cut

- Claim: $|\mathrm{MaxCut}(G)| = 2.|E(H)| - |\mathrm{MinVCover}(H)|$

Example 5: Max Cut

- Claim: $|MaxCut(G)| = 2.|E(H)| - |MinVCover(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V \backslash U + w)$ is a cut in G.

Example 5: Max Cut

- Claim: $|\mathrm{MaxCut}(G)| = 2.|E(H)| - |\mathrm{MinVCover}(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V \setminus U + w)$ is a cut in G.

- Let $S_G(U)$:= no. of edges in G with <u>exactly one</u> end vertex incident on a vertex in U.

Example 5: Max Cut

- Claim: $|MaxCut(G)| = 2.|E(H)| - |MinVCover(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V \backslash U + w)$ is a cut in G.

- Let $S_G(U)$ = no. of edges going out of U in G.

Example 5: Max Cut

- Claim: $|MaxCut(G)| = 2.|E(H)| - |MinVCover(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V \backslash U + w)$ is a cut in G.

- Let $S_G(U)$ = size of the cut $(U, V \backslash U + w)$.

Example 5: Max Cut

- Claim: $|\text{MaxCut}(G)| = 2.|E(H)| - |\text{MinVCover}(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V \backslash U + w)$ is a cut in G.

- Let $S_H(U)$:= no. of edges in H with <u>exactly one</u> end vertex incident on a vertex in U.

Example 5: Max Cut

- Claim: $|MaxCut(G)| = 2.|E(H)| - |MinVCover(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V\backslash U + w)$ is a cut in G.

- Then $S_G(U) = S_H(U) + \sum_{u \in U} (deg_H(u) - 1)$

$$= S_H(U) + \sum_{u \in U} deg_H(u) - |U|$$

Example 5: Max Cut

- Claim: $|MaxCut(G)| = 2.|E(H)| - |MinVCover(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V\backslash U + w)$ is a cut in G.

- Then $S_G(U) = S_H(U) + \sum_{u \in U} (deg_H(u) - 1)$

$$= S_H(U) + \sum_{u \in U} deg_H(u) - |U|$$

Obs: Twice the number of edges in H with <u>at least one</u> end vertex in U.

Example 5: Max Cut

- Claim: $|\mathrm{MaxCut}(G)| = 2.|E(H)| - |\mathrm{MinVCover}(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V \setminus U + w)$ is a cut in G.

- Then $S_G(U) = S_H(U) + \sum_{u \in U} (\deg_H(u) - 1)$

$$= S_H(U) + \sum_{u \in U} \deg_H(u) - |U|$$

$$= 2.|E_H(U)| - |U|$$

$E_H(U)$:= Set of edges in H with <u>at least one</u> end vertex in U.

Example 5: Max Cut

- Claim: $|MaxCut(G)| = 2.|E(H)| - |MinVCover(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V\backslash U + w)$ is a cut in G.

- Then $\boxed{S_G(U) = 2.|E_H(U)| - |U|}$ … Eqn (1)

- Proposition: If $(U, V\backslash U + w)$ is a <u>max cut</u> in G then U is a <u>vertex cover</u> in H.

Example 5: Max Cut

- Claim: $|\mathrm{MaxCut}(G)| = 2.|E(H)| - |\mathrm{MinVCover}(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V \setminus U + w)$ is a cut in G.

- Then $\boxed{S_G(U) = 2.|E_H(U)| - |U|}$ … Eqn (1)

- Proposition: If $(U, V \setminus U + w)$ is a <u>max cut</u> in G then U is a <u>vertex cover</u> in H.

  ⇒ $S_G(U) = |\mathrm{MaxCut}(G)| = 2.|E(H)| - |U|$

Example 5: Max Cut

- Claim: $|MaxCut(G)| = 2.|E(H)| - |MinVCover(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V \setminus U + w)$ is a cut in G.

- Then $\boxed{S_G(U) = 2.|E_H(U)| - |U|}$ … Eqn (1)

- Proposition: If $(U, V \setminus U + w)$ is a max cut in G then U is a vertex cover in H.

  U must be a minVCover in H

  $\Rightarrow$ $S_G(U) = |MaxCut(G)| = 2.|E(H)| - |U|$

Example 5: Max Cut

- Claim: $|\mathrm{MaxCut}(G)| = 2.|E(H)| - |\mathrm{MinVCover}(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V \setminus U + w)$ is a cut in G.

- Then $\boxed{S_G(U) = 2.|E_H(U)| - |U|}$ … Eqn (1)

- Proposition: If $(U, V \setminus U + w)$ is a <u>max cut</u> in G then U is a <u>vertex cover</u> in H.

  ⇒ $S_G(U) = |\mathrm{MaxCut}(G)| = 2.|E(H)| - |\mathrm{MinVCover}(H)|$

Example 5: Max Cut

- Claim: $|MaxCut(G)| = 2.|E(H)| - |MinVCover(H)|$
- Proof: Let $V(H) = V$. Then $V(G) = V + w$.

  Suppose $(U, V \backslash U + w)$ is a cut in G.

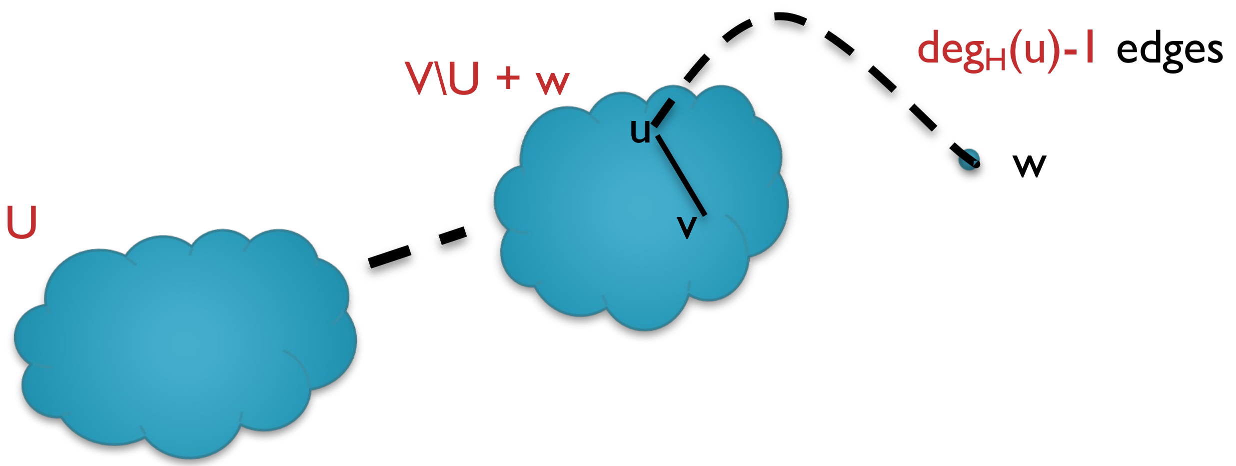
- Then $\boxed{S_G(U) = 2.|E_H(U)| - |U|}$ … Eqn (1)

- Proposition: If $(U, V \backslash U + w)$ is a <u>max cut</u> in G then U is a <u>vertex cover</u> in H.

Thus, the proof of the above claim follows from the proposition

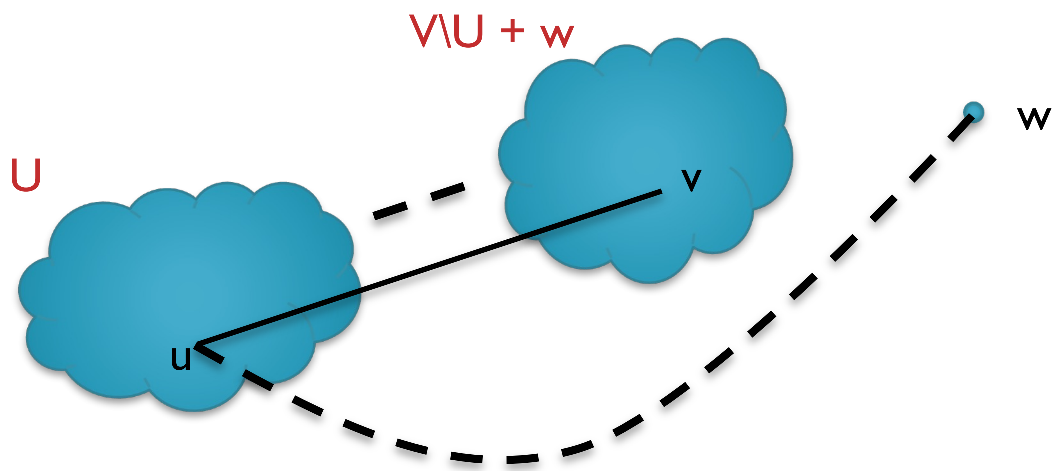
Example 5: Max Cut

- Proof of the Proposition: Suppose U is not a vertex cover



Example 5: Max Cut

- Proof of the Proposition: Suppose U is not a vertex cover



Gain: $\deg_H(u)-1 + 1$ edges.
Loss: At most $\deg_H(u)-1$ edges, these are the edges going from U to u.
Net gain: At least 1 edge. Hence the cut is not a max cut.

Search versus Decision

Search version of NP problems

- Recall: A language $L \subseteq \{0,1\}^*$ is in NP if
  - There's a *poly-time verifier* M and *poly. function* p s.t.
  - $x \in L$ iff there's a $u \in \{0,1\}^{p(|x|)}$ s.t $M(x, u) = 1$.

- Search version of L: Given an input $x \in \{0,1\}^*$, *find* a $u \in \{0,1\}^{p(|x|)}$ such that $M(x, u) = 1$, if such a u exists.

- Remark: Search version of L only makes sense once we have a verifier M in mind.

Search version of NP problems

- Recall: A language $L \subseteq \{0,1\}^*$ is in NP if
  - There's a *poly-time verifier* M and *poly. function* p s.t.
  - $x \in L$ iff there's a $u \in \{0,1\}^{p(|x|)}$ s.t $M(x, u) = 1$.

- Search version of L: Given an input $x \in \{0,1\}^*$, *find* a $u \in \{0,1\}^{p(|x|)}$ such that $M(x, u) = 1$, if such a u exists.

- Example: Given a 3CNF φ, find a satisfying assignment for φ if such an assignment exists.

Decision versus Search

- Is the search version of an NP-problem more difficult than the corresponding decision version?

Decision versus Search

- Is the search version of an NP-problem more difficult than the corresponding decision version?

- Theorem. Let $L \subseteq \{0,1\}^*$ be NP-complete. Then, the <u>search version of L</u> can be solved in poly-time <u>if and only if</u> the decision version can be solved in poly-time.

w.r.t any verifier M !

Decision versus Search

- Is the search version of an NP-problem more difficult than the corresponding decision version?

- Theorem. Let $L \subseteq \{0,1\}^*$ be NP-complete. Then, the search version of L can be solved in poly-time if and only if the decision version can be solved in poly-time.

- Proof. (search $\Longrightarrow$ decision) Obvious.

Decision versus Search

- Is the search version of an NP-problem more difficult than the corresponding decision version?

- Theorem. Let $L \subseteq \{0,1\}^*$ be NP-complete. Then, the search version of L can be solved in poly-time if and only if the decision version can be solved in poly-time.

- Proof. (decision $\Rightarrow$ search) We'll prove this for $L = SAT$ first.

SAT is *downward self-reducible*

- Proof. (decision $\Longrightarrow$ search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

SAT is *downward self-reducible*

- Proof. (decision $\Rightarrow$ search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$$\varphi(x_1,\ldots,x_n)$$

SAT is *downward self-reducible*

- Proof. (decision $\Rightarrow$ search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$\varphi(x_1,\ldots,x_n)$ $A(\varphi) = Y$

SAT is *downward self-reducible*

- Proof. (decision → search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$\varphi(x_1,\ldots,x_n)$ $A(\varphi) = Y$

$\varphi(0,\ldots,x_n)$

SAT is *downward self-reducible*

- Proof. (decision → search) Let L = SAT, and *A* be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$\varphi(x_1,\ldots,x_n)$ $A(\varphi) = Y$

$A(\ \varphi(0,..)\ ) = N$ $\varphi(0,\ldots,x_n)$

SAT is *downward self-reducible*

- Proof. (decision $\rightarrow$ search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$\varphi(x_1,\ldots,x_n)$ $A(\varphi) = Y$

$A(\ \varphi(0,..)\ ) = N$ $\varphi(0,\ldots,x_n)$ $\varphi(1,\ldots,x_n)$

SAT is *downward self-reducible*

- Proof. (decision $\longrightarrow$ search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$\varphi(x_1,\ldots,x_n)$ $A(\varphi)$ = Y

$A(\ \varphi(0,..)\ )$ = N $\varphi(0,\ldots,x_n)$

$\varphi(1,\ldots,x_n)$ $A(\ \varphi(1,..)\ )$ = Y

SAT is *downward self-reducible*

- **Proof.** (decision $\Rightarrow$ search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$\varphi(x_1,\ldots,x_n)$ A(φ) = Y

A(φ(0,..)) = N $\varphi(0,\ldots,x_n)$

$\varphi(1,\ldots,x_n)$ A(φ(1,..)) = Y

$\varphi(1,0,\ldots,x_n)$

SAT is *downward self-reducible*

- Proof. (decision $\longrightarrow$ search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$\varphi(x_1,\ldots,x_n)$ A(φ) = Y

A(φ(0,..)) = N $\varphi(0,\ldots,x_n)$

$\varphi(1,\ldots,x_n)$ A(φ(1,..)) = Y

A(φ(1,0,..)) = Y $\varphi(1,0,\ldots,x_n)$

SAT is *downward self-reducible*

- Proof. (decision $\Rightarrow$ search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$\varphi(x_1,\ldots,x_n)$ A(φ) = Y

A(φ(0,..)) = N $\varphi(0,\ldots,x_n)$

$\varphi(1,\ldots,x_n)$ A(φ(1,..)) = Y

A(φ(1,0,..)) = Y $\varphi(1,0,\ldots,x_n)$

$\varphi(1,0,0,\ldots,x_n)$

SAT is *downward self-reducible*

- **Proof.** (decision → search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$\varphi(x_1,\ldots,x_n)$ A(φ) = Y

A($\varphi(0,..)$) = N $\varphi(0,\ldots,x_n)$

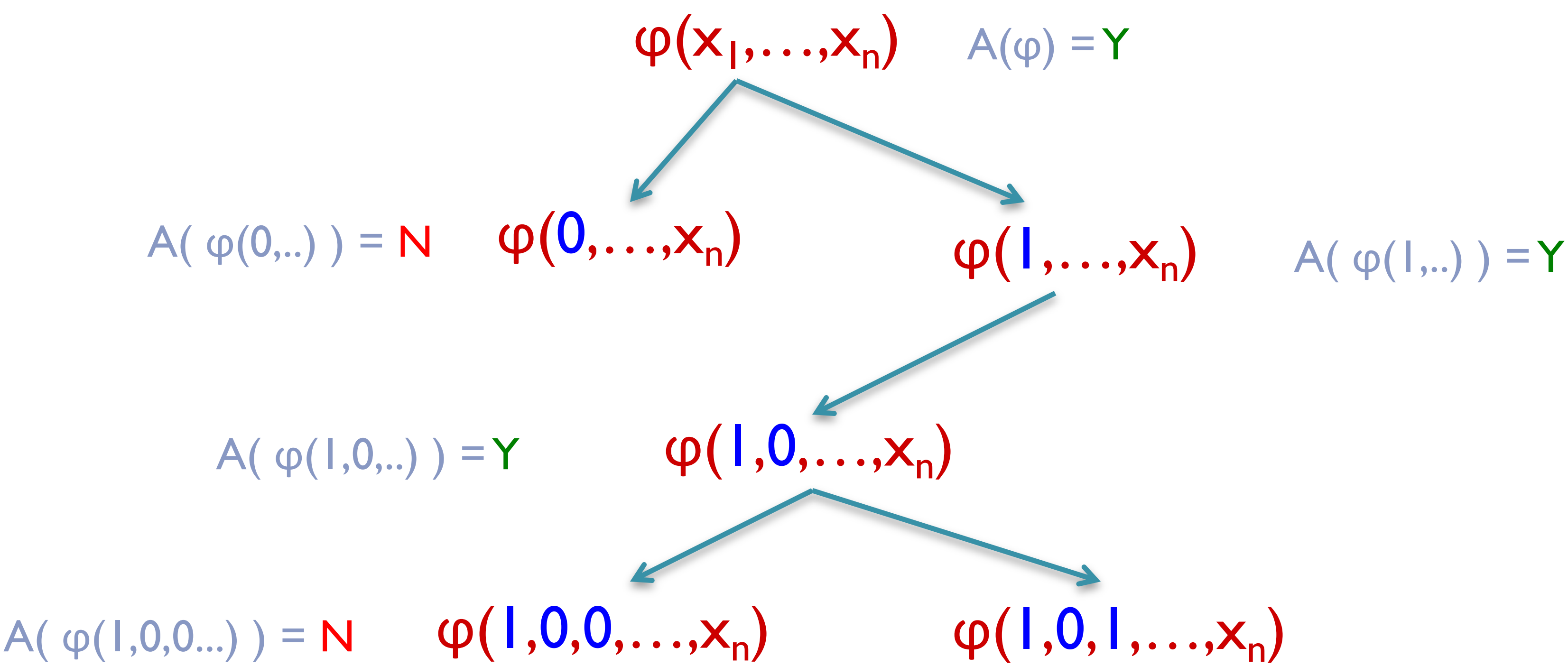
$\varphi(1,\ldots,x_n)$ A($\varphi(1,..)$) = Y

A($\varphi(1,0,..)$) = Y $\varphi(1,0,\ldots,x_n)$

A($\varphi(1,0,0...)$) = N $\varphi(1,0,0,\ldots,x_n)$

SAT is *downward self-reducible*

- Proof. (decision → search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.



SAT is *downward self-reducible*

- **Proof.** (decision → search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$\varphi(x_1,\ldots,x_n)$ A(φ) = Y

A(φ(0,..)) = N $\varphi(0,\ldots,x_n)$

$\varphi(1,\ldots,x_n)$ A(φ(1,..)) = Y

A(φ(1,0,..)) = Y $\varphi(1,0,\ldots,x_n)$

A(φ(1,0,0...)) = N $\varphi(1,0,0,\ldots,x_n)$

$\varphi(1,0,1,\ldots,x_n)$ A(φ(1,0,0...)) = Y

SAT is *downward self-reducible*

- Proof. (decision $\longrightarrow$ search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$\varphi(x_1,\ldots,x_n)$ A(φ) = Y

A(φ(0,..)) = N $\varphi(0,\ldots,x_n)$

$\varphi(1,\ldots,x_n)$ A(φ(1,..)) = Y

A(φ(1,0,..)) = Y $\varphi(1,0,\ldots,x_n)$

A(φ(1,0,0...)) = N $\varphi(1,0,0,\ldots,x_n)$

$\varphi(1,0,1,\ldots,x_n)$ A(φ(1,0,0...)) = Y

$\vdots$

SAT is *downward self-reducible*

- Proof. (decision $\longrightarrow$ search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

$\varphi(x_1,\ldots,x_n)$ A(φ) = Y

A(φ(0,..)) = N $\varphi(0,\ldots,x_n)$

$\varphi(1,\ldots,x_n)$ A(φ(1,..)) = Y

A(φ(1,0,..)) = Y $\varphi(1,0,\ldots,x_n)$

A(φ(1,0,0...)) = N $\varphi(1,0,0,\ldots,x_n)$

$\varphi(1,0,1,\ldots,x_n)$ A(φ(1,0,0...)) = Y

$\vdots$

SAT is *downward self-reducible*

- Proof. (decision $\Rightarrow$ search) Let L = SAT, and A be a poly-time algorithm to decide if $\varphi(x_1,\ldots,x_n)$ is satisfiable.

- We can find a satisfying assignment of φ with at most $2n$ calls to A.

Decision ≡ Search for NPC problems

- Proof. (decision $\Longrightarrow$ search) Let L be NP-complete, M be a verifier for L, and *B* be a poly-time algorithm to decide if $x \in L$.

Decision ≡ Search for NPC problems

- Proof. (decision ⟹ search) Let L be NP-complete, M be a verifier for L, and *B* be a poly-time algorithm to decide if $x \in L$.

SAT $\leq_p$ L

L $\leq_p$ SAT

Decision ≡ Search for NPC problems

- Proof. (decision $\Longrightarrow$ search) Let L be NP-complete, M be a verifier for L, and *B* be a poly-time algorithm to decide if $x \in L$.

$SAT \leq_p L$

$L \leq_p SAT$

$x \longmapsto \varphi_x$

Decision ≡ Search for NPC problems

- Proof. (decision ⟶ search) Let L be NP-complete, M be a verifier for L, and *B* be a poly-time algorithm to decide if $x \in L$.

$\text{SAT} \le_p L$

$L \le_p \text{SAT}$

$x \longmapsto \varphi_x$

Important note:

From Cook-Levin theorem, we can find a certificate of $x \in L$ (w.r.t. M) from a satisfying assignment of φ_x.

Decision ≡ Search for NPC problems

- Proof. (decision ⟹ search) Let L be NP-complete, M be a verifier for L, and *B* be a poly-time algorithm to decide if $x \in L$.

$\mathrm{SAT} \leq_p L$ $\qquad\qquad$ $L \leq_p \mathrm{SAT}$

$$x \longmapsto \varphi_x$$

How to find a satisfying assignment for φ_x <u>using algorithm B</u> ?

Decision ≡ Search for NPC problems

- Proof. (decision ⟶ search) Let L be NP-complete, M be a verifier for L, and *B* be a poly-time algorithm to decide if $x \in L$.

$\text{SAT} \leq_p L$

$L \leq_p \text{SAT}$

$x \longmapsto \varphi_x$

How to find a satisfying assignment for φ_x using algorithm B ?

...we know how using *A*, which is a poly-time decider for SAT

Decision ≡ Search for NPC problems

- Proof. (decision ⟶ search) Let L be NP-complete, M be a verifier for L, and *B* be a poly-time algorithm to decide if $x \in L$.

$$\text{SAT} \le_p L \qquad\qquad L \le_p \text{SAT}$$

$$\varphi \longmapsto f(\varphi) \qquad\qquad x \longmapsto \varphi_x$$

How to find a satisfying assignment for φ_x using algorithm B ?

...we know how using *A*, which is a poly-time decider for SAT

Take $A(\varphi) = B(\, f(\varphi)\,)$.

Decision versus Search

- Is *search* equivalent to *decision* for every NP problem?

- Graph Isomorphism (GI) is in NP and (we'll see later that) it is unlikely to be NP-complete.
- Yet, the natural search version of GI reduces in polynomial-time to the decision version *(homework)*.

Decision versus Search

- Is *search* equivalent to *decision* for every NP problem?

Probably not!

Decision versus Search

- Is *search* equivalent to *decision* for every NP problem?

- Let $EE = \bigcup_{c \geq 0} DTIME\,(2^{c.2^n})$ and

  $NEE = \bigcup_{c \geq 0} NTIME\,(2^{c.2^n})$

  Doubly exponential analogues of P and NP

- Class NTIME(T(n)) will be defined formally in the next lecture.

Decision versus Search

- Is *search* equivalent to *decision* for every NP problem?

- Theorem. *(Bellare & Goldwasser 1994)* If EE $\neq$ NEE then there's a language in NP for which search does not reduce to decision.

Decision versus Search

- Is *search* equivalent to *decision* for every NP problem?

- Theorem. *(Bellare & Goldwasser 1994)* If EE $\neq$ NEE then there's a language in NP for which search does not reduce to decision.

- Checking if a number n is **composite** can be done in polynomial-time, but finding a factor of n is not known to be solvable in polynomial-time.
- We'll show that Intfact is unlikely to be NP-complete.

Decision versus Search

- Is *search* equivalent to *decision* for every NP problem?

- Theorem. *(Bellare & Goldwasser 1994)* If EE ≠ NEE then there's a language in NP for which search does not reduce to decision.

- Sometimes, the decision version of a problem can be trivial but the search version is possibly hard. E.g., Computing Nash Equilibrium (see class PPAD).

Homework: Read about **total NP functions**