Computational Complexity Theory

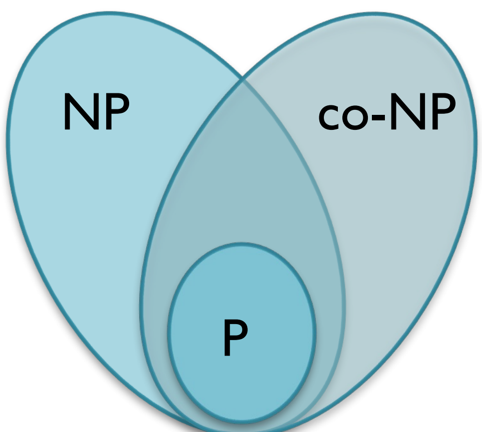
Lecture 6: Diagonalization; Time Hierarchy; Ladner's theorem

Department of Computer Science,
Indian Institute of Science

Recap: Class co-NP

- **Definition.** For every $L \subseteq \{0,1\}^*$ let $\overline{L} = \{0,1\}^* \setminus L$.
  A language L is in co-NP if $\overline{L}$ is in NP.

- **Example.** $\overline{SAT} = \{\varphi : \varphi$ is *not* satisfiable$\}$.



Recap: Alternate definition of co-NP

- Recall, a language $L \subseteq \{0,1\}^*$ is in NP if there's a *poly-time verifier* M such that

$$x \in L \iff \exists u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x,u) = 1$$

$$x \in \bar{L} \iff \forall u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x,u) = 0$$

$$x \in \bar{L} \iff \forall u \in \{0,1\}^{p(|x|)} \text{ s.t. } \bar{M}(x,u) = 1$$

- Definition. A language $L \subseteq \{0,1\}^*$ is in co-NP if there's a polynomial function p and a *poly-time* TM M such that

$$x \in L \iff \forall u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x,u) = 1$$

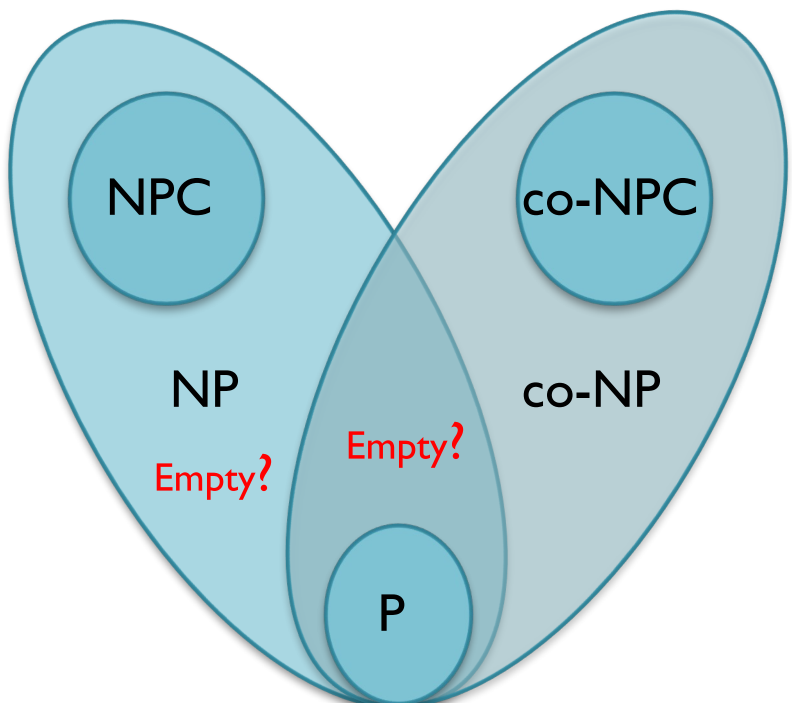
↑ for NP this was ∃

Recap: co-NP-completeness

- Definition. A language L' $\subseteq \{0,1\}^*$ is co-NP-complete if
  - L' is in co-NP
  - Every language L in co-NP is polynomial-time (Karp) reducible to L'.

- Theorem. $\overline{\text{SAT}}$ and TAUTOLOGY are co-NP-complete.

Recap: The diagram again



If an NP-complete language belongs to co-NP then

NP ⊆ co-NP

⇒ NP = co-NP

Let C_1 and C_2 be two complexity classes.

If $C_1 \subseteq C_2$, then co-$C_1 \subseteq$ co-C_2.

Obs. co-(co-C) = C .

Recap: FACT in NP ∩ co-NP

- Integer factoring.

  FACT = {(N, U): there's a prime in [U] dividing N}

- Claim. FACT ∈ NP ∩ co-NP

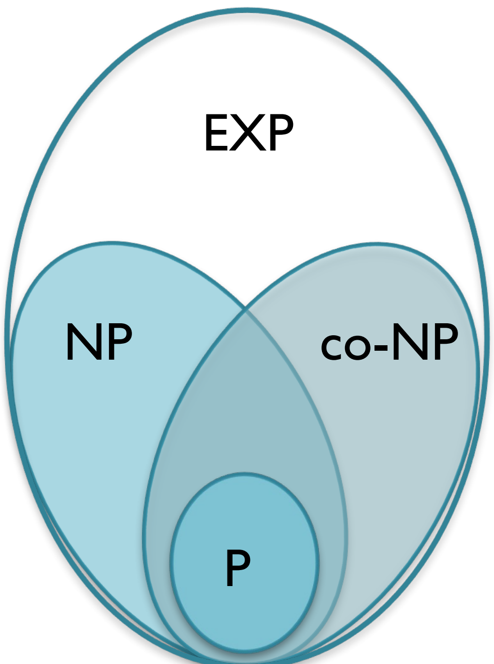
- So, FACT is NP-complete implies NP = co-NP.

Recap: Class EXP

- **Definition.** Class EXP is the exponential time analogue of class P.

$$EXP = \bigcup_{c \geq 1} DTIME(2^{n^c})$$

- **Observation.** $P \subseteq NP \subseteq EXP$



Recap: Class EXP

- Definition. Class EXP is the exponential time analogue of class P.

$$EXP = \bigcup_{c \geq 1} DTIME(2^{n^c})$$

- Observation. P $\subseteq$ NP $\subseteq$ EXP

- *Exponential Time Hypothesis.* *(Impagliazzo & Paturi 1999)* Any algorithm for 3-SAT takes $\geq 2^{\delta . n}$ time, where $\delta > 0$ is some fixed constant and n is the no. of variables.

In other words, δ cannot be made arbitrarily close to 0.

Recap: Class EXP

- Definition. Class EXP is the exponential time analogue of class P.

$$EXP = \bigcup_{c \geq 1} DTIME(2^{n^c})$$

We'll address this using ***diagonalization***

Is $P \subsetneq EXP$?

- Observation. $P \subseteq NP \subseteq EXP$

- *Exponential Time Hypothesis.* *(Impagliazzo & Paturi 1999)* Any algorithm for 3-SAT takes $\geq 2^{\delta.n}$ time, where $\delta > 0$ is some fixed constant and n is the no. of variables.

Diagonalization

Diagonalization

- *Diagonalization* refers to a class of techniques used in complexity theory to separate complexity classes.

Diagonalization

- *Diagonalization* refers to a class of techniques used in complexity theory to separate complexity classes.
- These techniques are characterized by <u>two</u> main features:

Diagonalization

- *Diagonalization* refers to a class of techniques used in complexity theory to separate complexity classes.
- These techniques are characterized by <u>two</u> main features:
  1. There's a universal TM U that when given strings α and x, simulates M_α on x with only a <u>*small*</u> overhead.

Diagonalization

- *Diagonalization* refers to a class of techniques used in complexity theory to separate complexity classes.
- These techniques are characterized by two main features:

  1. There's a universal TM U that when given strings α and x, simulates M_α on x with only a *small* overhead.

     If M_α takes T time on x then U takes $O(T \log T)$ time to simulate M_α on x.

Diagonalization

- *Diagonalization* refers to a class of techniques used in complexity theory to separate complexity classes.
- These techniques are characterized by two main features:
    1. There's a universal TM U that when given strings α and x, simulates M_α on x with only a *small* overhead.
    2. Every string represents some TM, and every TM can be represented by *infinitely many* strings.

Time Hierarchy Theorem

- An application of Diagonalization

Time Hierarchy Theorem

- Let f(n) and g(n) be <u>time-constructible</u> functions s.t.,

  $$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. *(Hartmanis & Stearns 1965)*

  $$\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$$

- Theorem. $\mathrm{P} \subsetneq \mathrm{EXP}$

- This type of results are called <u>**lower bounds**</u>.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

Task: Show that there's a language L decided by a TM D with time complexity $O(n^2)$ s.t., any TM M with runtime $O(n)$ cannot decide L.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

  TM D :

  1. On input x, compute $|x|^2$.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$

Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

TM D :

1. On input x, compute $|x|^2$.
2. Simulate M_x on x for $|x|^2$ steps.

D's time steps not M_x's time steps.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

  $$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

  TM D :

  1. On input x, compute $|x|^2$.
  2. Simulate M_x on x for $|x|^2$ steps.
     a. If M_x stops and outputs b then output $1-b$.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

  TM D :

  1. On input x, compute $|x|^2$.
  2. Simulate M_x on x for $|x|^2$ steps.
     a. If M_x stops and outputs b then output $1-b$.
     b. Otherwise, output 1.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

  TM D :

  1. On input x, compute $|x|^2$.
  2. Simulate M_x on x for $|x|^2$ steps.
     a. If M_x stops and outputs b then output 1-b.
     b. Otherwise, output 1.

D outputs the **opposite** of what M_x outputs.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

  D runs in $O(n^2)$ time as n^2 is <u>time-constructible</u>.

Time Hierarchy Theorem

- Let f(n) and g(n) be time-constructible functions s.t.,

  $$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

  Claim. There's no TM M with running time O(n) that decides L (the language accepted by D).

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \cdot \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

➢ <u>For contradiction</u>, suppose M decides L and runs for at most $c.n$ steps on inputs of length n. (i.e., $M(x) = D(x)$ for all x)

Time Hierarchy Theorem

- Let f(n) and g(n) be time-constructible functions s.t.,

$$f(n) . \log f(n) = o(g(n)).$$

- Theorem. $\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

  - For contradiction, suppose M decides L and runs for at most c.n steps on inputs of length n. (i.e., M(x) = D(x) for all x)

c is a constant

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$

Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

  - ➢ For contradiction, suppose M decides L and runs for at most $c.n$ steps on inputs of length n. (i.e., $M(x) = D(x)$ for all x)
  - ➢ Think of a <u>sufficiently large</u> x such that $M = M_x$.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

  - For contradiction, suppose M decides L and runs for at most $c.n$ steps on inputs of length n. (i.e., $M(x) = D(x)$ for all x)
  - Think of a sufficiently large x such that $M = M_x$.
  - Suppose $M(x) = M_x(x) = b$.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

  - For contradiction, suppose M decides L and runs for at most $c.n$ steps on inputs of length n. (i.e., $M(x) = D(x)$ for all x)
  - Think of a <u>sufficiently large</u> x such that $M = M_x$.
  - Suppose $M(x) = M_x(x) = b$.
  - D on input x, simulates M_x on x for $|x|^2$ steps.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

- ➢ For contradiction, suppose M decides L and runs for at most $c.n$ steps on inputs of length n. (i.e., $M(x) = D(x)$ for all x)
- ➢ Think of a <u>sufficiently large</u> x such that $M = M_x$.
- ➢ Suppose $M(x) = M_x(x) = b$.
- ➢ D on input x, simulates M_x on x for $|x|^2$ steps. Since M_x stops within $c.|x|$ steps, D's simulation also stops within $c'.c.\,|x|.\log|x|$ steps.

↑ c' is a constant

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.
  - For contradiction, suppose M decides L and runs for at most $c.n$ steps on inputs of length n. (i.e., $M(x) = D(x)$ for all x)
  - Think of a sufficiently large x such that $M = M_x$.
  - Suppose $M(x) = M_x(x) = b$.
  - D on input x, simulates M_x on x for $|x|^2$ steps. Since M_x stops within $c.|x|$ steps, D's simulation also stops within $c'.c. |x|. \log |x|$ steps. (as $c'.c. |x|. \log |x| < |x|^2$ for sufficiently large x)

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

- For contradiction, suppose M decides L and runs for at most $c.n$ steps on inputs of length n. (i.e., $M(x) = D(x)$ for all x)
- Think of a <u>sufficiently large</u> x such that $M = M_x$.
- Suppose $M(x) = M_x(x) = b$.
- D on input x, simulates M_x on x for $|x|^2$ steps. Since M_x stops within $c.|x|$ steps, D's simulation also stops within $c'.c. |x|. \log |x|$ steps. And D outputs the **opposite** of what M_x outputs!

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

  - For contradiction, suppose M decides L and runs for at most $c.n$ steps on inputs of length n. (i.e., $M(x) = D(x)$ for all x)
  - Think of a sufficiently large x such that $M = M_x$.
  - Suppose $M(x) = M_x(x) = b$.
  - Hence, $D(x) = 1\text{-}b$.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

  Proof. We'll prove with $f(n) = n$ and $g(n) = n^2$.

  - For contradiction, suppose M decides L and runs for at most $c.n$ steps on inputs of length n. (i.e., $M(x) = D(x)$ for all x)
  - Think of a <u>sufficiently large</u> x such that $M = M_x$.
  - Suppose $M(x) = M_x(x) = b$.
  - Hence, $D(x) = 1\text{-}b$.

Contradiction! M does not decide L.

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

  $$f(n) \cdot \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

- Theorem. $P \subsetneq EXP$

  Proof. Similar (homework)

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

$$f(n) \cdot \log f(n) = o(g(n)).$$

- Theorem. $DTIME(f(n)) \subsetneq DTIME(g(n))$

- Theorem. $P \subsetneq EXP$
- **No** EXP-complete problem (under poly-time Karp reduction) is in P.

E.g., Decide if a TM halts in k steps; generalized versions of games such as chess, checkers, Go, etc.

Time Hierarchy Theorem

- Is there a natural problem that takes close to n^2 time?

Time Hierarchy Theorem

- Is there a natural problem that takes close to n^2 time?

- 3SUM: Given a list of n numbers, check if there exists 3 numbers in the list that sum to zero.

Time Hierarchy Theorem

- Is there a natural problem that takes close to n^2 time?

- 3SUM: Given a list of n numbers, check if there exists 3 numbers in the list that sum to zero.
- Conjecture. ***No*** algorithm solves 3SUM in $O(n^{2-\varepsilon})$ time for some constant $\varepsilon > 0$.

Time Hierarchy Theorem

- Is there a natural problem that takes close to n^2 time?

- 3SUM: Given a list of n numbers, check if there exists 3 numbers in the list that sum to zero.
- Conjecture. ***No*** algorithm solves 3SUM in $O(n^{2-\varepsilon})$ time for some constant $\varepsilon > 0$.

- However, there's a $\sim O(n^2 / (\log n)^2)$ time algorithm for 3SUM. ("~" suppressing a poly(log log n) factor.)

Time Hierarchy Theorem

- Is there a natural problem that takes close to n^2 time?

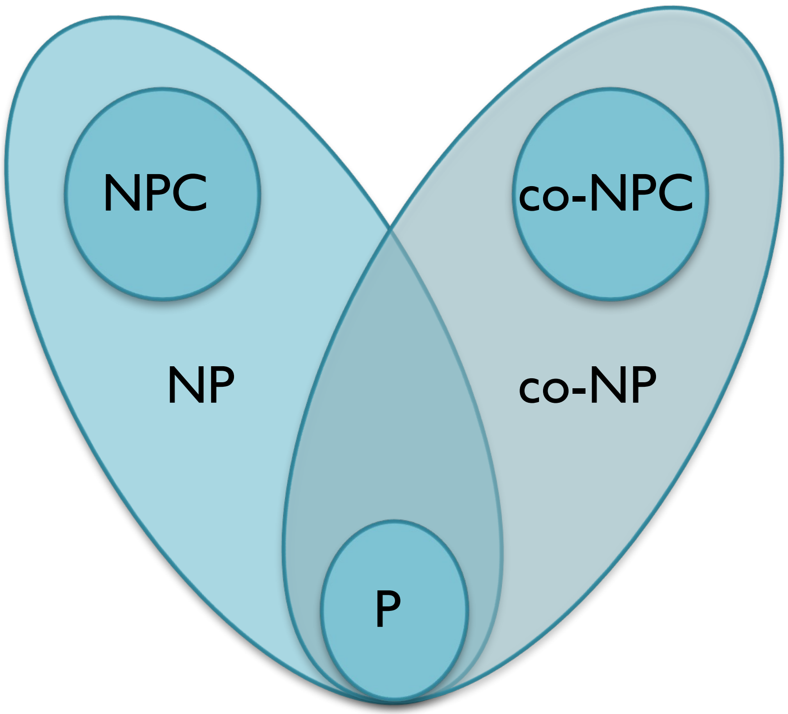
- 3SUM: Given a list of n numbers, check if there exists 3 numbers in the list that sum to zero.
- Conjecture. ***No*** algorithm solves 3SUM in $O(n^{2-\varepsilon})$ time for some constant $\varepsilon > 0$.
- kSUM: Given a list of n numbers, check if there exists k numbers in the list that sum to zero.

Time Hierarchy Theorem

- Is there a natural problem that takes close to n^2 time?

- 3SUM: Given a list of n numbers, check if there exists 3 numbers in the list that sum to zero.
- Conjecture. ***No*** algorithm solves 3SUM in $O(n^{2-\varepsilon})$ time for some constant $\varepsilon > 0$.
- kSUM: Given a list of n numbers, check if there exists k numbers in the list that sum to zero.
- Theorem *(Patrascu & Williams 2010)*. ETH implies kSUM requires $n^{\Omega(k)}$ time.

Revisiting NP∩co-NP



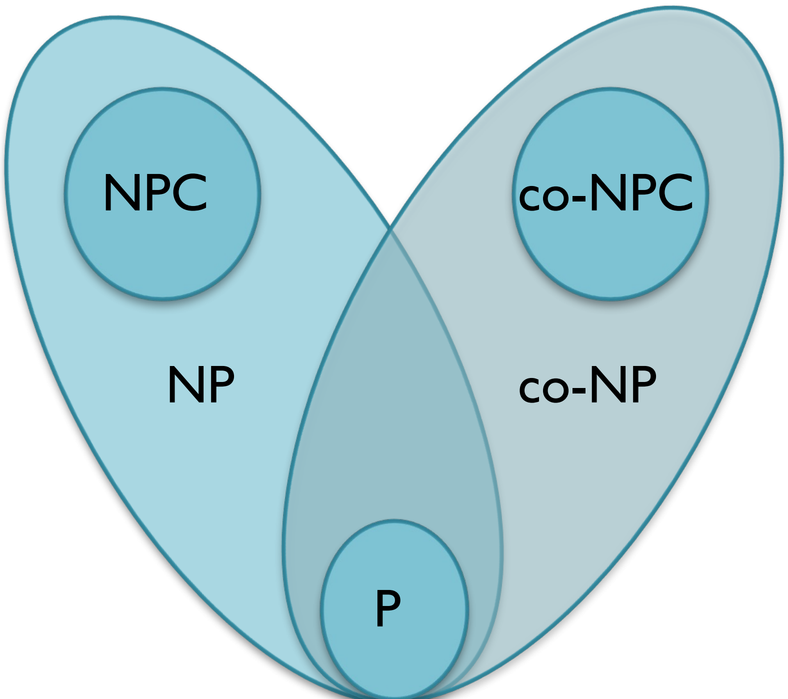
Conjecture: NP ≠ co-NP

⇓

P ≠ NP

General belief: P ≠ NP ∩ co-NP

Revisiting NP∩co-NP



Conjecture: NP ≠ co-NP

⬇

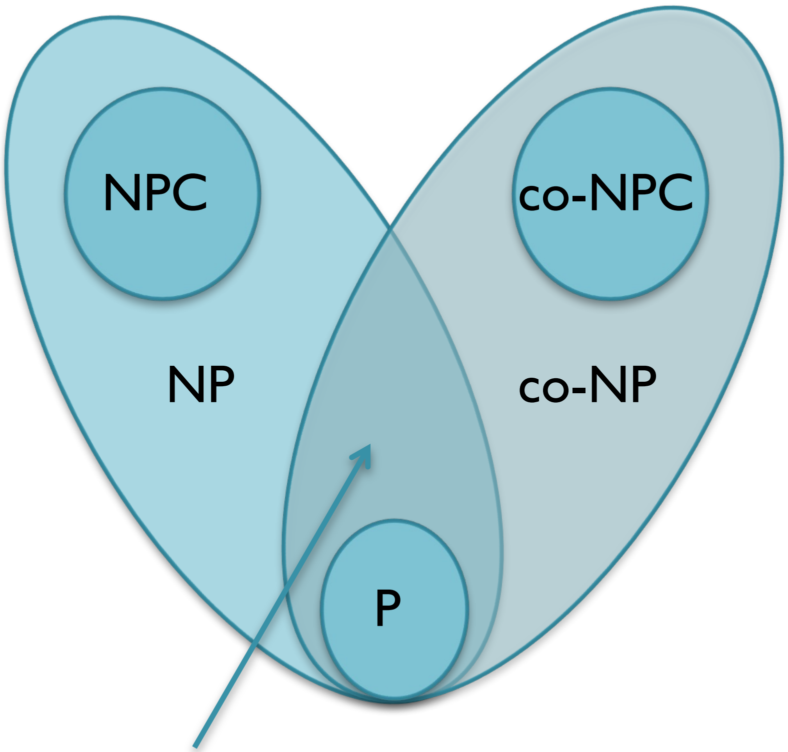
P ≠ NP

General belief: P ≠ NP ∩ co-NP

Edmonds (1966)

…conjectured P = NP ∩ co-NP

Revisiting NP∩co-NP



Conjecture: NP ≠ co-NP

⇓

P ≠ NP

General belief: P ≠ NP ∩ co-NP

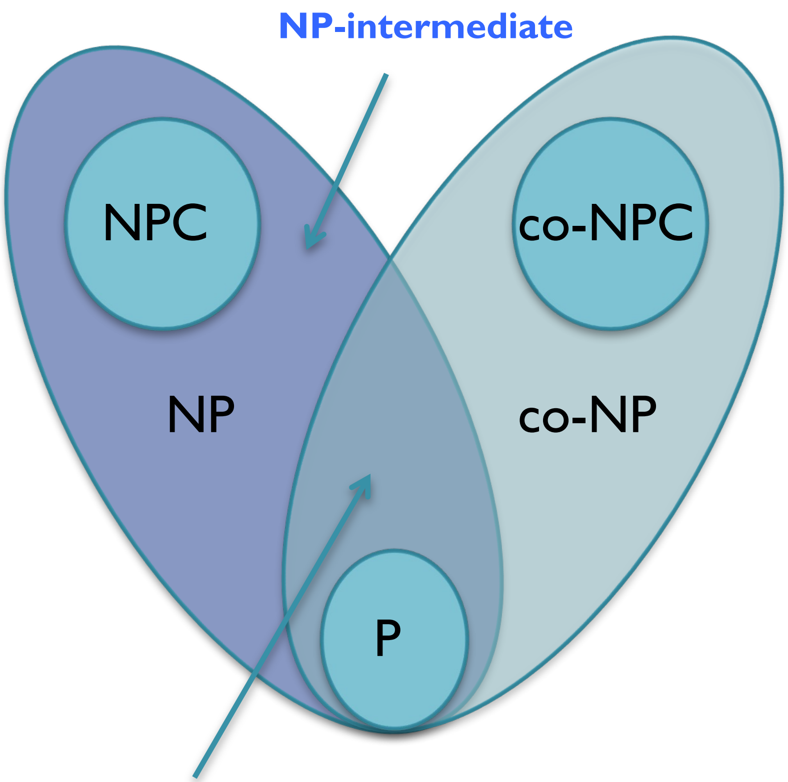
Check:
https://cstheory.stackexchange.com/questions/20021/reasons-to-believe-p-ne-np-cap-conp-or-not

Check if the shortest non-zero vector in an n-dimensional lattice has length at most 1 or at least √n.

- Integer factoring (FACT)
- Approximate shortest vector in a lattice

Ref: "*Lattice problems in NP∩co-NP*" by Aharonov & Regev (2005)

NP-intermediate problems



Conjecture: NP ≠ co-NP

⬇

P ≠ NP

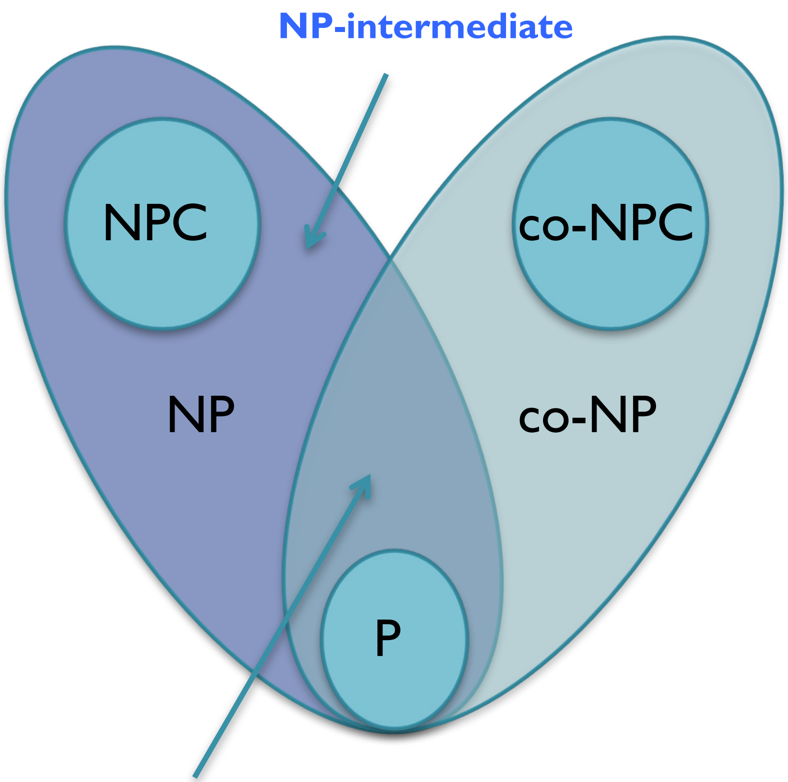
General belief: P ≠ NP ∩ co-NP

Obs: If NP ≠ co-NP and FACT ∉ P then FACT is NP-intermediate.

- Integer factoring (FACT)
- Approximate shortest vector in a lattice

Ref: "*Lattice problems in NP∩co-NP*" by Aharonov & Regev (2005)

NP-intermediate problems



Conjecture: NP ≠ co-NP

⇓

P ≠ NP

General belief: P ≠ NP ∩ co-NP

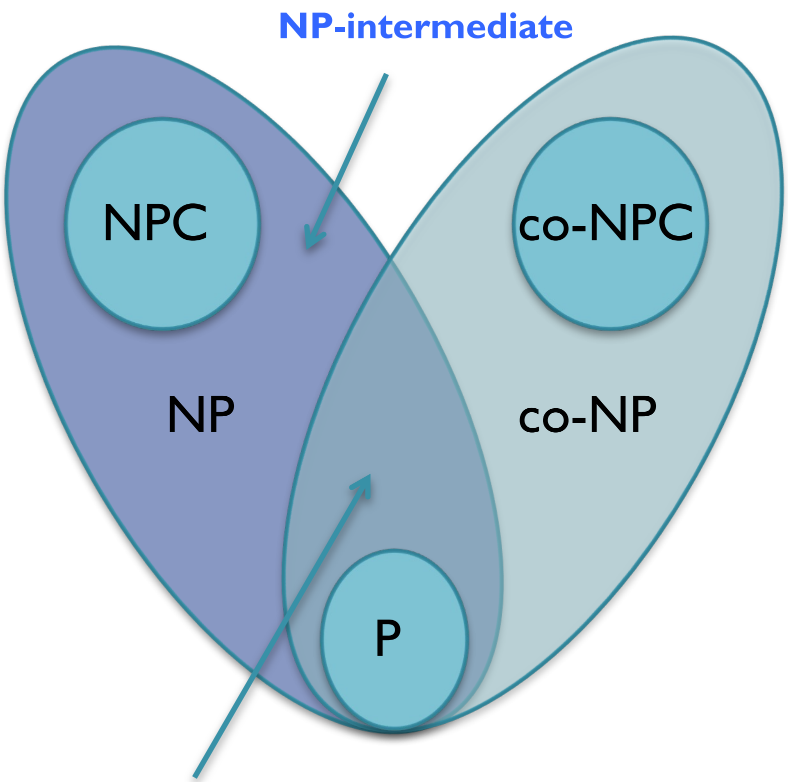
Obs: If NP ≠ co-NP and FACT ∉ P then FACT is NP-intermediate.

Ladner's theorem: P ≠ NP implies existence of a NP-intermediate language.

- Integer factoring (FACT)
- Approximate shortest vector in a lattice

Ref: "*Lattice problems in NP∩co-NP*" by Aharonov & Regev (2005)

NP-intermediate problems



- Integer factoring (FACT)
- Approximate shortest vector in a lattice

Ref: "*Lattice problems in NP∩co-NP*" by Aharonov & Regev (2005)

Conjecture: NP ≠ co-NP

⬇

P ≠ NP

General belief: P ≠ NP ∩ co-NP

Obs: If NP ≠ co-NP and FACT ∉ P then FACT is NP-intermediate.

Ladner's theorem: P ≠ NP implies existence of a NP-intermediate language.

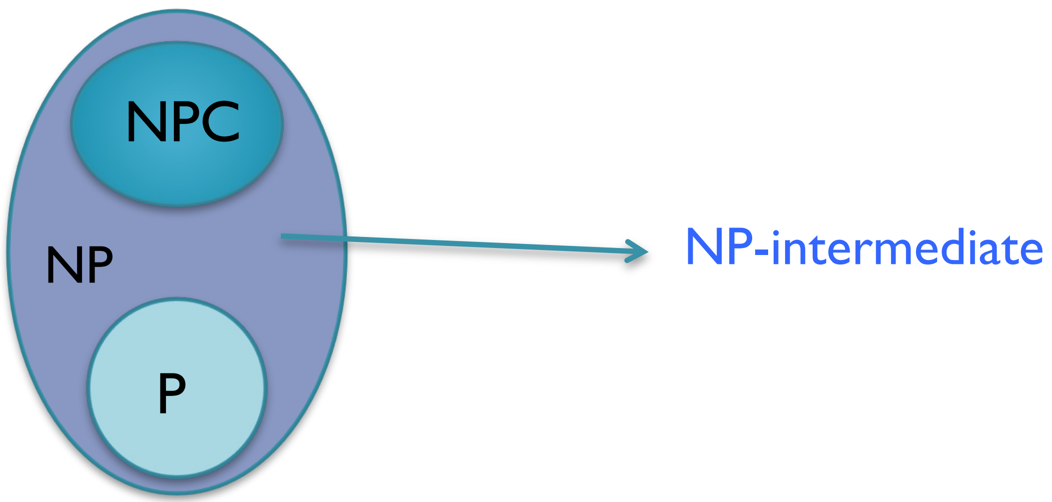
(proved using ***diagonalization***)

Ladner's Theorem

- Another application of Diagonalization

NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.



NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.

- Theorem. *(Ladner 1975)* If P ≠ NP then there is a NP-intermediate language.

NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.

- Theorem. *(Ladner 1975)* If P ≠ NP then there is a NP-intermediate language.

  Proof. A delicate argument using diagonalization.

NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.

- Theorem. *(Ladner 1975)* If $P \neq NP$ then there is a NP-intermediate language.

  Proof. Let $H: \mathbb{N} \to \mathbb{N}$ be a function.

NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.

- Theorem. *(Ladner 1975)* If P ≠ NP then there is a NP-intermediate language.

  Proof. Let $H: \mathbb{N} \rightarrow \mathbb{N}$ be a function.

  Let $SAT_H = \{\Psi 0 1^{m^{H(m)}} : \Psi \in SAT \text{ and } |\Psi| = m\}$

NP-intermediate problems

- **Definition.** A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.

- **Theorem.** *(Ladner 1975)* If $P \neq NP$ then there is a NP-intermediate language.

  Proof. Let $H: \mathbb{N} \rightarrow \mathbb{N}$ be a function.

  Let $SAT_H = \{\Psi 0\, 1^{m^{H(m)}} : \Psi \in SAT \text{ and } |\Psi| = m\}$

  H would be defined in such a way that SAT_H is NP-intermediate (assuming $P \neq NP$)

Ladner's theorem: Constructing H

- Theorem. There's a function $H: N \rightarrow N$ such that

  1. $H(m)$ is computable from m in $O(m^3)$ time.

Ladner's theorem: Constructing H

- Theorem. There's a function $H: \mathbb{N} \to \mathbb{N}$ such that

1. $H(m)$ is computable from m in $O(m^3)$ time.

2. If $SAT_H \in P$ then $H(m) \leq C$ (a constant).

for every m

Ladner's theorem: Constructing H

- Theorem. There's a function $H: \mathbb{N} \rightarrow \mathbb{N}$ such that

  1. $H(m)$ is computable from m in $O(m^3)$ time.

  2. If $SAT_H \in P$ then $H(m) \leq C$ (a constant).

  3. If $SAT_H \notin P$ then $H(m) \rightarrow \infty$ with m.

Ladner's theorem: Constructing H

- Theorem. There's a function $H: N \rightarrow N$ such that

  1. $H(m)$ is computable from m in $O(m^3)$ time.

  2. If $SAT_H \in P$ then $H(m) \leq C$ (a constant).

  3. If $SAT_H \notin P$ then $H(m) \rightarrow \infty$ with m.

Proof: Later (uses diagonalization).

Let's see the proof of Ladner's theorem
assuming the existence of such a "special" H.

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.

Ladner's theorem: Proof

$P \neq NP$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.
- This implies a poly-time algorithm for SAT as follows:

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.
- This implies a poly-time algorithm for SAT as follows:
  - On input φ , find $m = |\varphi|$.

Ladner's theorem: Proof

$P \neq NP$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.
- This implies a poly-time algorithm for SAT as follows:
  - On input φ, find $m = |\varphi|$.
  - Compute $H(m)$, and construct the string $\varphi\ 0\ 1^{m^{H(m)}}$

Ladner's theorem: Proof

$P \neq NP$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.
- This implies a poly-time algorithm for SAT as follows:
  - On input φ , find $m = |\varphi|$.
  - Compute $H(m)$, and construct the string $\varphi\, 0\, 1^{m^{H(m)}}$
  - Check if $\varphi\, 0\, 1^{m^{H(m)}}$ belongs to SAT_H .

Ladner's theorem: Proof

$P \neq NP$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.
- This implies a poly-time algorithm for SAT as follows:
  - On input φ, find $m = |\varphi|$.
  - Compute $H(m)$, and construct the string $\varphi\, 0\, 1^{m^{H(m)}}$
  - Check if $\varphi\, 0\, 1^{m^{H(m)}}$ belongs to SAT_H.

length at most $m + 1 + m^C$

Ladner's theorem: Proof

$P \neq NP$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.
- This implies a poly-time algorithm for SAT as follows:
  - On input φ, find $m = |\varphi|$.
  - Compute $H(m)$, and construct the string $\varphi\, 0\, 1^{m^{H(m)}}$
  - Check if $\varphi\, 0\, 1^{m^{H(m)}}$ belongs to SAT_H.
- As $P \neq NP$, it must be that $SAT_H \notin P$.

Ladner's theorem: Proof

$P \neq NP$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$SAT \leq_p SAT_H$ $\qquad\qquad$ $\varphi \xmapsto{f} \psi\, 0\, 1^k$

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \varphi \overset{f}{\longmapsto} \Psi\, 0\, 1^k$$

$$|\varphi| = n \qquad |\Psi\, 0\, 1^k| = n^c$$

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H$$

$$\varphi \overset{f}{\longmapsto} \psi\ 0\ 1^k$$

$|\varphi| = n$ $|\psi\ 0\ 1^k| = n^c$

Let m_0 be the largest s.t. $H(m_0) \leq 2c$.

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \varphi \overset{f}{\longmapsto} \Psi\, 0\, 1^k$$

Let m_0 be the largest s.t. $H(m_0) \leq 2c$.

- On input φ, compute $f(\varphi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \varphi \overset{f}{\longmapsto} \Psi \; 0 \; 1^k$$

Let m_0 be the largest s.t. $H(m_0) \leq 2c$.

- On input φ, compute $f(\varphi) = \Psi \; 0 \; 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$SAT \leq_p SAT_H$ $\qquad$ $\varphi \overset{f}{\longmapsto} \Psi\ 0\ 1^k$

Let m_0 be the largest s.t. $H(m_0) \leq 2c$.

- On input φ, compute $f(\varphi) = \Psi\ 0\ 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$. (Homework: Verify that this can be done in poly(n) time.)

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \to \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$SAT \leq_p SAT_H$ $\varphi \overset{f}{\longmapsto} \Psi\ 0\ 1^k$

Let m_0 be the largest s.t. $H(m_0) \leq 2c$.

- On input φ, compute $f(\varphi) = \Psi\ 0\ 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.

Either $m \leq m_0$ (in which case the task reduces to checking if a constant-size Ψ is satisfiable),

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$SAT \leq_p SAT_H$ $\qquad\qquad$ $\varphi \stackrel{f}{\longmapsto} \Psi\ 0\ 1^k$

Let m_0 be the largest s.t. $H(m_0) \leq 2c$.

- On input φ, compute $f(\varphi) = \Psi\ 0\ 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.

or $H(m) > 2c$ (as $H(m)$ tends to infinity with m).

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad\qquad \varphi \overset{f}{\longmapsto} \psi\, 0\, 1^k$$

➢ On input φ, compute $f(\varphi) = \psi\, 0\, 1^k$. Let $m = |\psi|$.

➢ Compute $H(m)$ and check if $k = m^{H(m)}$.

➢ Hence, w.l.o.g. $\qquad |f(\varphi)| \geq k > m^{2c}$

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad\qquad \varphi \xrightarrow{f} \Psi\ 0\ 1^k$$

- On input φ, compute $f(\varphi) = \Psi\ 0\ 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.
- Hence, w.l.o.g. $n^c = |f(\varphi)| \geq k > m^{2c}$

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad\qquad \varphi \overset{f}{\longmapsto} \Psi\ 0\ 1^k$$

- On input φ, compute $f(\varphi) = \Psi\ 0\ 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.
- Hence, $\sqrt{n} \geq m$.

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad\qquad \varphi \xmapsto{f} \psi\, 0\, 1^k$$

- On input φ, compute $f(\varphi) = \psi\, 0\, 1^k$. Let $m = |\psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.
- Hence, $\sqrt{n} \geq m$. Also $\varphi \in SAT$ iff $\psi \in SAT$

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$SAT \leq_p SAT_H$ $\qquad\qquad$ $\varphi \stackrel{f}{\longmapsto} \Psi\, 0\, 1^k$

- On input φ, compute $f(\varphi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.
- Hence, $\sqrt{n} \geq m$. Also $\varphi \in SAT$ iff $\Psi \in SAT$

Thus, checking if an n-size formula φ is satisfiable reduces to checking if a $\sqrt{n}$-size formula Ψ is satisfiable.

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \rightarrow \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$SAT \leq_p SAT_H$ $\qquad\qquad$ $\varphi \overset{f}{\longmapsto} \Psi\, 0\, 1^k$

  - On input φ, compute $f(\varphi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
  - Compute $H(m)$ and check if $k = m^{H(m)}$.
  - Hence, $\sqrt{n} \geq m$. Also $\varphi \in SAT$ iff $\Psi \in SAT$

Do this recursively! Only $O(\log \log n)$ recursive steps required.

Ladner's theorem: Proof

$$P \neq NP$$

- Suppose SAT_H is NP-complete. Then $H(m) \to \infty$ with m.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad\qquad \varphi \overset{f}{\longmapsto} \Psi\ 0\ 1^k$$

  - On input φ, compute $f(\varphi) = \Psi\ 0\ 1^k$. Let $m = |\Psi|$.
  - Compute $H(m)$ and check if $k = m^{H(m)}$.
  - Hence, $\sqrt{n} \geq m$. Also $\varphi \in SAT$ iff $\Psi \in SAT$

- Hence SAT_H is not NP-complete, as $P \neq NP$.

Ladner's theorem: Properties of H

- Theorem. There's a function $H: \mathbb{N} \rightarrow \mathbb{N}$ such that

  1. $H(m)$ is computable from m in $O(m^3)$ time.
  2. If $SAT_H \in P$ then $H(m) \leq C$ (a constant).
  3. If $SAT_H \notin P$ then $H(m) \rightarrow \infty$ with m.

- $SAT_H = \{\Psi 0 1^{m^{H(m)}} : \Psi \in SAT$ and $|\Psi| = m\}$