Computational Complexity Theory

Lecture 8: Space complexity classes; Savitch's theorem

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Space bounded computation

Space bounded computation

- Here, we are interested to find out how much of *work space* is required to solve a problem.
- For convenience, think of TMs with a separate read-only input tape and one or more work tapes. Work space is the number of cells in the work tapes of a TM M visited by M's heads during a computation.

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- Definition. Let $S: \mathbb{N} \to \mathbb{N}$ be a function. A language L is in DSPACE(S(n)) if there's a TM M that decides L using O(S(n)) work space on inputs of length n.

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- Definition. Let $S: \mathbb{N} \rightarrow \mathbb{N}$ be a function. A language L is in NSPACE(S(n)) if there's a NTM M that decides L using O(S(n)) work space on inputs of length n, regardless of M's nondeterministic choices.

Space bounded computation

- We'll refer to 'work space' as 'space'. For convenience, assume there's a single work tape.
- If the output has many bits, then we will assume that the TM has a separate write-only output tape.

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- If the output has many bits, then we will assume that the TM has a separate write-only output tape.

- Definition. Let $S: \mathbb{N} \longrightarrow \mathbb{N}$ be a function. S is *space constructible* if $S(n) \geq \log n$ and there's a TM that computes $S(|x|)$ from x using $O(S(|x|))$ space.

Relation between time and space

- Obs. DTIME(S(n)) $\subsetneq$ DSPACE(S(n)) $\subseteq$ NSPACE(S(n)).

Hopcroft, Paul & Valiant 1977

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- Theorem *(Williams 2025)*. For all functions $T(n) \geq n$, every multitape Turing machine running in time T can be simulated in space only $O(\sqrt{T \log T})$.

Relation between time and space

- Obs. DTIME(S(n)) $\subsetneq$ DSPACE(S(n)) $\subseteq$ NSPACE(S(n)).

- Theorem. NSPACE(S(n)) $\subseteq$ DTIME($2^{O(S(n))}$), if S is space constructible.
- Proof. Uses the notion of *configuration graph* of a TM. We'll see this shortly.

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- Definition.

$$\mathrm{L} = \mathrm{DSPACE}(\log n)$$

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Giving space at least log n gives a TM at least the power to remember the index of a cell.

Relation between time and space

- **Obs.** DTIME(S(n)) $\subsetneq$ DSPACE(S(n)) $\subseteq$ NSPACE(S(n)).
- **Theorem.** NSPACE(S(n)) $\subseteq$ DTIME($2^{O(S(n))}$), if S is space constructible.
- **Note.** *Williams'* theorem does not imply P $\subsetneq$ PSPACE although it's an important step in this direction.
- **Open.** Is P $\neq$ PSPACE ?

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- Theorem. $NSPACE(S(n)) \subseteq DTIME(2^{O(S(n))})$, if S is space constructible.

- Theorem. $L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP$

Follows from the above theorem

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- Theorem. $\mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} \subseteq \mathrm{EXP}$

Run through all possible choices of certificates of the verifier and **reuse** space.

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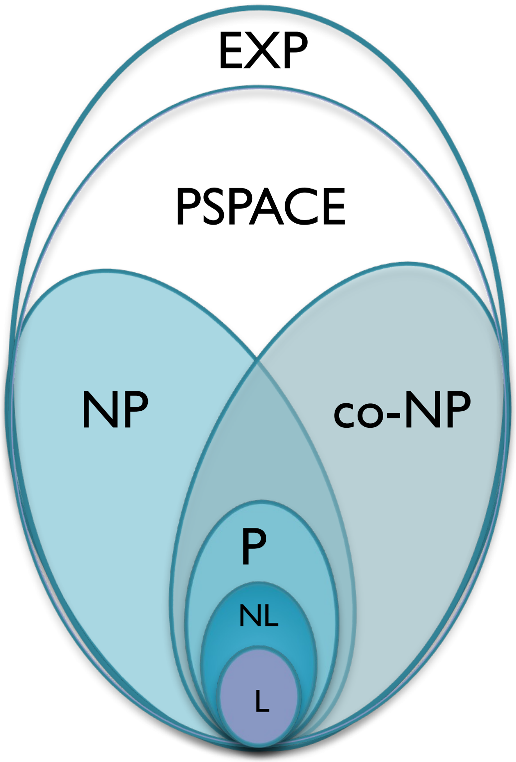

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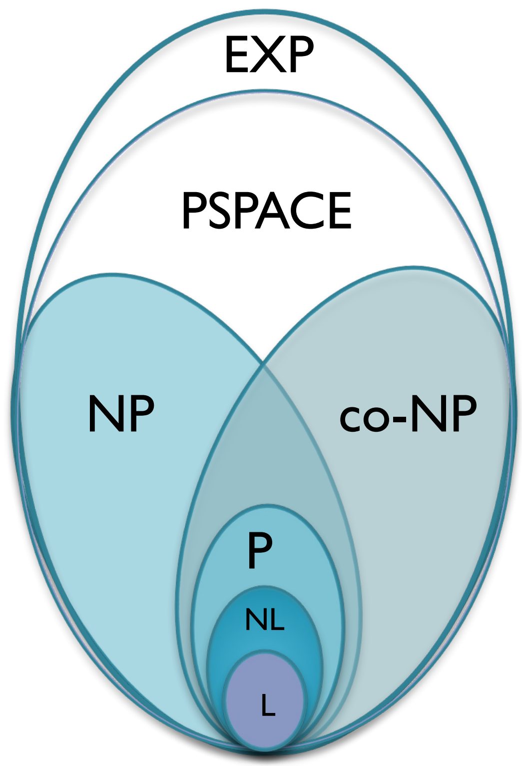
Homework: Integer addition and multiplication are in (functional) L.

Integer division is also in (functional) L. *(Chiu, Davida & Litow 2001)*



Configuration graph

- Definition. A *configuration* of a TM M on input x, at any particular step of its execution, consists of

    (a) the nonblank symbols of its work tapes,

    (b) the current state,

    (c) the current head positions.

It captures a 'snapshot' of M at any particular moment of execution.

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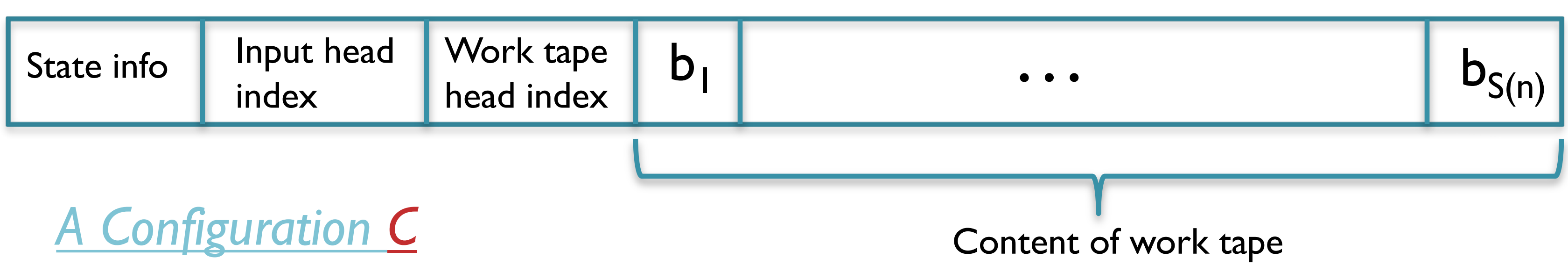

A Configuration C

Configuration graph

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  (a) the nonblank symbols of its work tapes,
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  (c) the current head positions.

It captures a 'snapshot' of M at any particular moment of execution.

State info	Input head index	Work tape head index	b_1	. . .	$b_{S(n)}$

Note: A configuration C can be represented using $O(S(n))$ bits if M uses $S(n) = \Omega(\log n)$ space on n-bit inputs.

Configuration graph

- Definition. A *configuration graph* of a TM M on input x, denoted $G_{M,x}$, is a directed graph whose nodes are all the possible configurations of M on input x. There's an edge from one configuration C_1 to another C_2, if C_2 can be reached from C_1 by an application of M's transition function(s).

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- Number of nodes in $G_{M,x} = 2^{O(S(n))}$, if M uses $S(n)$ space on n-bit inputs

Configuration graph

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- If M is a DTM then every node C in $G_{M,x}$ has at most one outgoing edge. If M is an NTM then every node C in $G_{M,x}$ has at most two outgoing edges.

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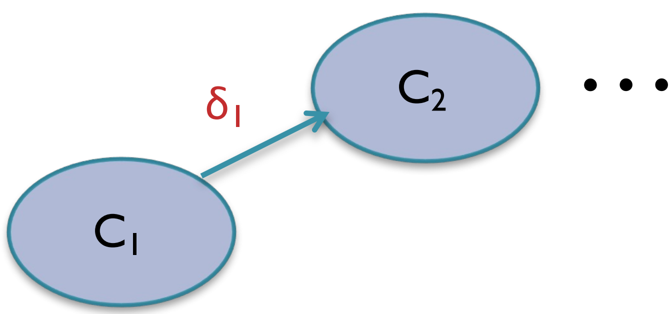
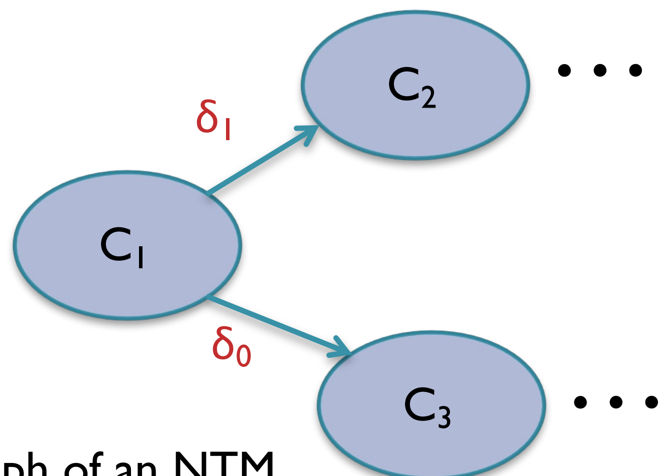


Conf. graph of a DTM



Conf. graph of an NTM

Configuration graph

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- By erasing the contents of the work tape at the end, bringing the head at the beginning, and having a q_{accept} state, we can assume that there's a unique C_{accept} configuration. Configuration C_{start} is well defined.

Configuration graph

- Definition. A *configuration graph* of a TM M on input x, denoted $G_{M,x}$, is a directed graph whose nodes are all the possible configurations of M on input x. There's an edge from one configuration C_1 to another C_2, if C_2 can be reached from C_1 by an application of M's transition function(s).

- M accepts x if and only if there's a path from C_{start} to C_{accept} in $G_{M,x}$.

Relation between time and space

- Obs. $\mathrm{DTIME}(S(n)) \subsetneq \mathrm{DSPACE}(S(n)) \subseteq \mathrm{NSPACE}(S(n))$.
- Theorem. $\mathrm{NSPACE}(S(n)) \subseteq \mathrm{DTIME}(2^{O(S(n))})$, if S is space constructible.
- Proof. Let $L \in \mathrm{NSPACE}(S(n))$ and M be an NTM deciding L using $O(S(n))$ space on length n inputs.
- On input x, compute the configuration graph $G_{M,x}$ of M and check if there's a **path** from C_{start} to C_{accept}. Running time is $2^{O(S(n))}$.

Natural problems?

- Definition.

$$L = DSPACE(\log n)$$

$$NL = NSPACE(\log n)$$

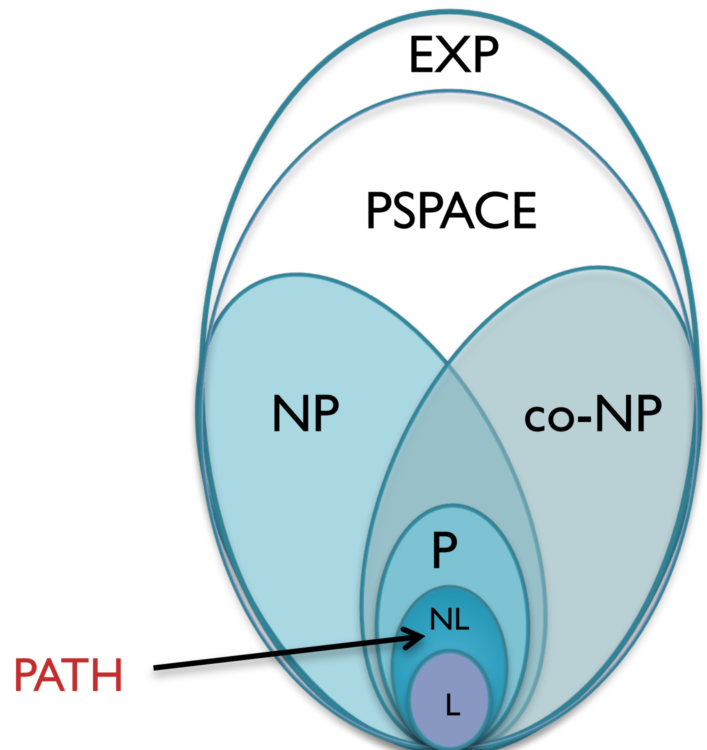
$$PSPACE = \bigcup_{c>0} DSPACE(n^c)$$

- Theorem. $L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP$.

- Are there natural problems in L, NL and PSPACE ?

PATH: A canonical problem in NL

- PATH = {(G,s,t) : G is a directed graph having a path from s to t}.
- Obs. PATH is in NL.



PATH: A canonical problem in NL

- PATH = {(G,s,t) : G is a directed graph having a path from s to t}.
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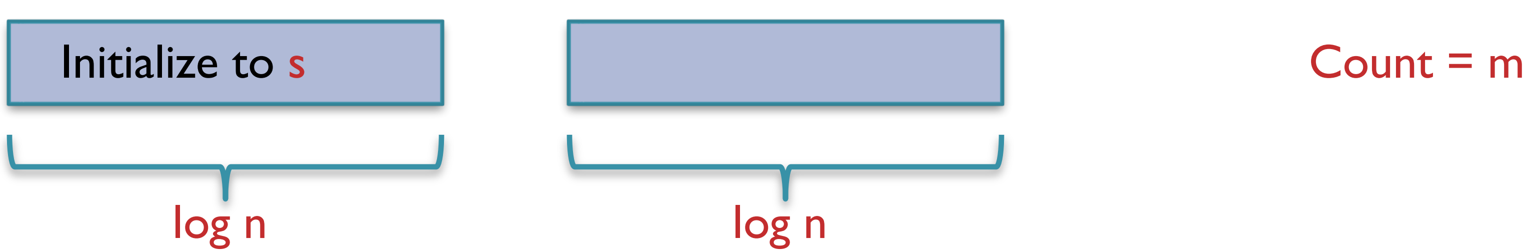
- Proof. Count the no. of vertices in G, let it be n. Set aside two memory locations of $\log n$ bits each. Initialize a counter, say Count $= m < n$.



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Initialize to s

Guess a vertex v_1

Count = m

If there's a edge from s to v_1, decrease count by 1. Else o/p 0 and stop.

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- Proof. Count the no. of vertices in G, let it be n. Set aside two memory locations of log n bits each. Initialize a counter, say Count = m < n.

Set to v_1

Guess a vertex v_2

Count = m-1

If there's a edge from v_1 to v_2, decrease count by 1. Else o/p 0 and stop.

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- Obs. PATH is in NL.
- Proof. Count the no. of vertices in G, let it be n. Set aside two memory locations of log n bits each. Initialize a counter, say Count = m < n.

Set to v_2

Guess a vertex v_3

Count = m-2

If there's a edge from v_2 to v_3, decrease count by 1. Else o/p 0 and stop.

…and so on.

PATH: A canonical problem in NL

- PATH = {(G,s,t) : G is a directed graph having a path from s to t}.
- Obs. PATH is in NL.

- Proof. Count the no. of vertices in G, let it be n. Set aside two memory locations of log n bits each. Initialize a counter, say Count = m < n.

Set to v_{m-1}

Set to t

Count = 1

If there's a edge from v_{m-1} to t, o/p 1 and stop.
Else o/p 0 and stop.

PATH: A canonical problem in NL

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Set to t

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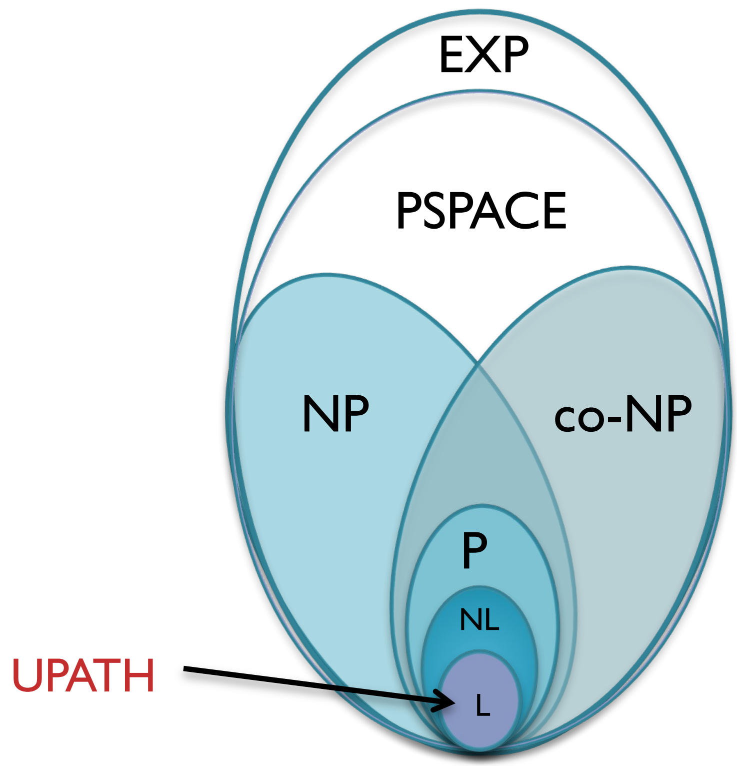
Space complexity = O(log n)

UPATH: A problem in L

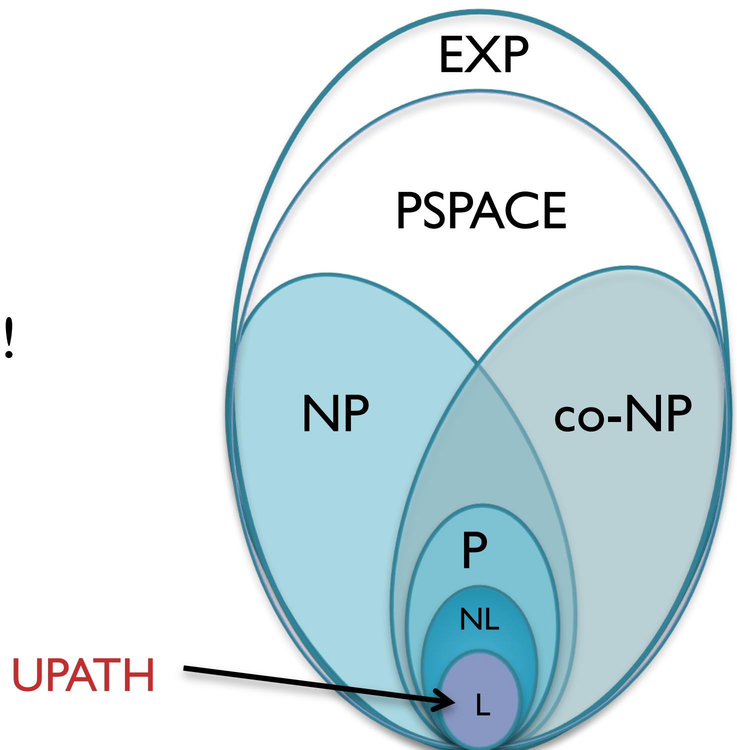
- UPATH = {(G,s,t) : G is an undirected graph having a path from s to t}.
- Theorem *(Reingold 2005)*. UPATH is in L.



UPATH: A problem in L

- UPATH = {(G,s,t) : G is an undirected graph having a path from s to t}.
- Theorem *(Reingold 2005)*. UPATH is in L.

Is PATH in L ?
If yes, then L = NL !
(will prove later)



Space Hierarchy Theorem

- Theorem. *(Stearns, Hartmanis & Lewis 1965)* If f and g are space-constructible functions and $f(n) = o(g(n))$, then $\mathrm{SPACE}(f(n)) \subsetneq \mathrm{SPACE}(g(n))$.

- Proof. Homework.

- Theorem. $\mathrm{L} \subsetneq \mathrm{PSPACE}$.

PSPACE = NPSPACE

Savitch's theorem

- Theorem. $\mathrm{NSPACE}(S(n)) \subseteq \mathrm{DSPACE}(S(n)^2)$, where $S(n)$ is space constructible. (So, PSPACE = NPSPACE)

- Proof. Let $L \in \mathrm{NSPACE}(S(n))$, and M be an NTM requiring $O(S(n))$ space to decide L. We'll show that there's a TM N requiring $O(S(n)^2)$ space to decide L.

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- On input x, N checks if there's a path from C_{start} to C_{accept} in $G_{M,x}$ as follows: Let |x| = n.

Savitch's theorem

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- Proof. (contd.) N computes m = O(S(n)), the no. of bits required to represent a configuration of M. It also finds out C_{start} and C_{accept}. Then N checks if there's a path from C_{start} to C_{accept} of length at most 2^m in $G_{M,x}$ recursively using the following procedure.

- REACH(C_1, C_2, i) : returns 1 if there's a path from C_1 to C_2 of length at most 2^i in $G_{M,x}$; 0 otherwise.

Savitch's theorem

- Theorem. NSPACE(S(n)) $\subseteq$ DSPACE(S(n)^2), where S(n) is space constructible. (So, PSPACE = NPSPACE)

Space constructibility of S(n) used here

- Proof. (contd.) N computes m = O(S(n)), the no. of bits required to represent a configuration of M. It also finds out C_{start} and C_{accept}. Then N checks if there's a path from C_{start} to C_{accept} of length at most 2^m in $G_{M,x}$ recursively using the following procedure.

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- Proof.
- REACH(C_1, C_2, i) {

    If $i = 0$ check if C_1 and C_2 are adjacent.

    Else, for every configurations C,

    a_1 = REACH(C_1, C, i-1)

    a_2 = REACH(C, C_2, i-1)

    if $a_1=1$ & $a_2=1$, return 1. Else return 0.

}

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Require O(S(n)) space

Savitch's theorem

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  If $i = 0$ check if C_1 and C_2 are adjacent.

  Else, for every configurations C,

  a_1 = REACH(C_1, C, i-1)
  a_2 = REACH(C, C_2, i-1)
  } Reuse space

  if a_1=1 & a_2=1, return 1. Else return 0.

}

Savitch's theorem

- Theorem. $NSPACE(S(n)) \subseteq DSPACE(S(n)^2)$, where $S(n)$ is space constructible. (So, PSPACE = NPSPACE)

- Proof.

$$Space(i) = Space(i-1) + O(S(n))$$

- Space complexity: $O(S(n)^2)$

Savitch's theorem

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- Proof.

$$\text{Space}(i) = \text{Space}(i-1) + O(S(n))$$

- Space complexity: $O(S(n)^2)$

$$\text{Time}(i) = 2^m.2.\text{Time}(i-1) + O(S(n))$$

- Time complexity: $2^{O(S(n)^2)}$

Savitch's theorem

- Theorem. $NSPACE(S(n)) \subseteq DSPACE(S(n)^2)$, where $S(n)$ is space constructible. (So, PSPACE = NPSPACE)

- Proof.

$$Space(i) = Space(i-1) + O(S(n))$$

- Space complexity: $O(S(n)^2)$

$$Time(i) = 2^m.2.Time(i-1) + O(S(n))$$

- Time complexity: $2^{O(S(n)^2)}$

Recall, $NSPACE(S(n)) \subseteq DTIME(2^{O(S(n))})$. There's an algorithm with time complexity $2^{O(S(n))}$, but higher space requirement.