Computational Complexity Theory

Lecture 9: PSPACE-completeness; Log-space reductions; NL-completeness

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Recap: Space bounded computation

- Here, we are interested to find out how much of *work space* is required to solve a problem.
- For convenience, think of TMs with a separate read-only input tape and one or more work tapes. Work space is the number of cells in the work tapes of a TM M visited by M's heads during a computation.

- Definition. Let $S: \mathbb{N} \rightarrow \mathbb{N}$ be a function. A language L is in DSPACE(S(n)) if there's a TM M that decides L using O(S(n)) work space on inputs of length n.

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- Definition. Let $S: \mathbb{N} \rightarrow \mathbb{N}$ be a function. A language L is in NSPACE(S(n)) if there's a NTM M that decides L using O(S(n)) work space on inputs of length n, regardless of M's nondeterministic choices.

Recap: Space bounded computation

- We'll refer to 'work space' as 'space'. For convenience, assume there's a single work tape.
- If the output has many bits, then we will assume that the TM has a separate write-only output tape.

- Definition. Let $S: \mathbb{N} \longrightarrow \mathbb{N}$ be a function. S is *space constructible* if $S(n) \geq \log n$ and there's a TM that computes $S(|x|)$ from x using $O(S(|x|))$ space.

Recap: Time and space

- Obs. $\mathrm{DTIME}(S(n)) \subsetneq \mathrm{DSPACE}(S(n)) \subseteq \mathrm{NSPACE}(S(n))$.

- Theorem. $\mathrm{NSPACE}(S(n)) \subseteq \mathrm{DTIME}(2^{O(S(n))})$, if S is space constructible.

- Definition.

$$L = \mathrm{DSPACE}(\log n)$$

$$NL = \mathrm{NSPACE}(\log n)$$

$$\mathrm{PSPACE} = \bigcup_{c > 0} \mathrm{DSPACE}(n^c)$$

Giving space at least log n gives a TM at least the power to remember the index of a cell.

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$$\mathrm{PSPACE} = \bigcup_{c > 0} \mathrm{DSPACE}(n^c)$$

Why did we not define NPSPACE?
We saw that unlike P and NP,
PSPACE = NPSPACE

Recap: Time and space

- Obs. DTIME(S(n)) $\subsetneq$ DSPACE(S(n)) $\subseteq$ NSPACE(S(n)).
- Theorem. NSPACE(S(n)) $\subseteq$ DTIME($2^{O(S(n))}$), if S is space constructible.
- Caution. The *Hopcroft-Paul-Valiant* theorem does not imply P $\subsetneq$ PSPACE.
- Open. Is P $\neq$ PSPACE ?

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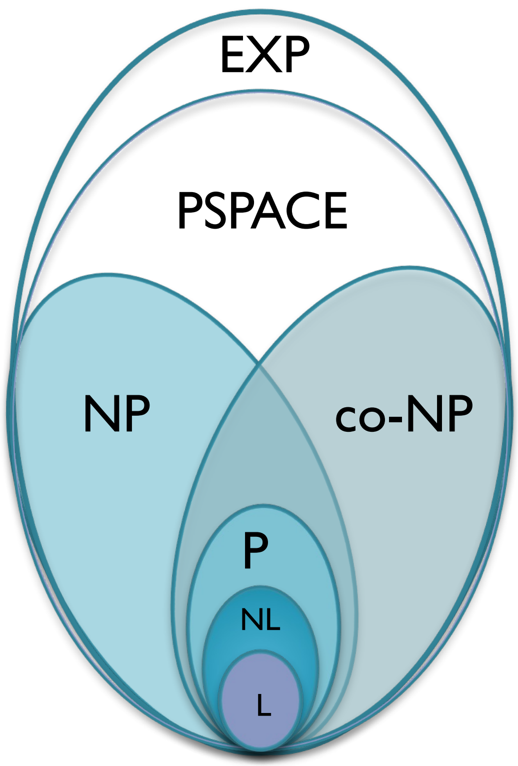
- Theorem. NSPACE(S(n)) $\subseteq$ DTIME($2^{O(S(n))}$), if S is space constructible.

Homework: Integer addition and multiplication are in (functional) L.

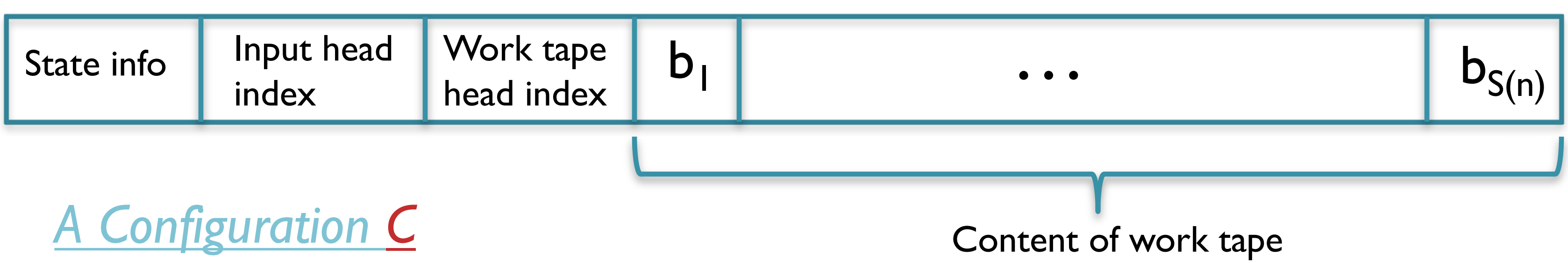
Integer division is also in (functional) L. *(Chiu, Davida & Litow 2001)*



Recap: Configuration graph

- Definition. A *configuration* of a TM M on input x, at any particular step of its execution, consists of
  - (a) the nonblank symbols of its work tapes,
  - (b) the current state,
  - (c) the current head positions.

It captures a 'snapshot' of M at any particular moment of execution.



A Configuration C

Recap: Configuration graph

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  (a) the nonblank symbols of its work tapes,

  (b) the current state,

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It captures a 'snapshot' of M at any particular moment of execution.

State info	Input head index	Work tape head index	b_1	. . .	$b_{S(n)}$

Note: A configuration C can be represented using $O(S(n))$ bits if M uses $S(n) = \Omega(\log n)$ space on n-bit inputs.

Recap: Configuration graph

- Definition. A *configuration graph* of a TM M on input x, denoted $G_{M,x}$, is a directed graph whose nodes are all the possible configurations of M on input x. There's an edge from one configuration C_1 to another C_2, if C_2 can be reached from C_1 by an application of M's transition function(s).

- Number of nodes in $G_{M,x} = 2^{O(S(n))}$, if M uses $S(n)$ space on n-bit inputs

Recap: Configuration graph

- Definition. A *configuration graph* of a TM M on input x, denoted $G_{M,x}$, is a directed graph whose nodes are all the possible configurations of M on input x. There's an edge from one configuration C_1 to another C_2, if C_2 can be reached from C_1 by an application of M's transition function(s).

- If M is a DTM then every node C in $G_{M,x}$ has at most one outgoing edge. If M is an NTM then every node C in $G_{M,x}$ has at most two outgoing edges.

Recap: Time and space

- Obs. DTIME(S(n)) $\subsetneq$ DSPACE(S(n)) $\subseteq$ NSPACE(S(n)).

- Theorem. NSPACE(S(n)) $\subseteq$ DTIME($2^{O(S(n))}$), if S is space constructible.

- Proof. Let L $\in$ NSPACE(S(n)) and M be an NTM deciding L using O(S(n)) space on length n inputs.
- On input x, compute the configuration graph $G_{M,x}$ of M and check if there's a **path** from C_{start} to C_{accept} . Running time is $2^{O(S(n))}$.

Recap: Natural problems?

- Definition.

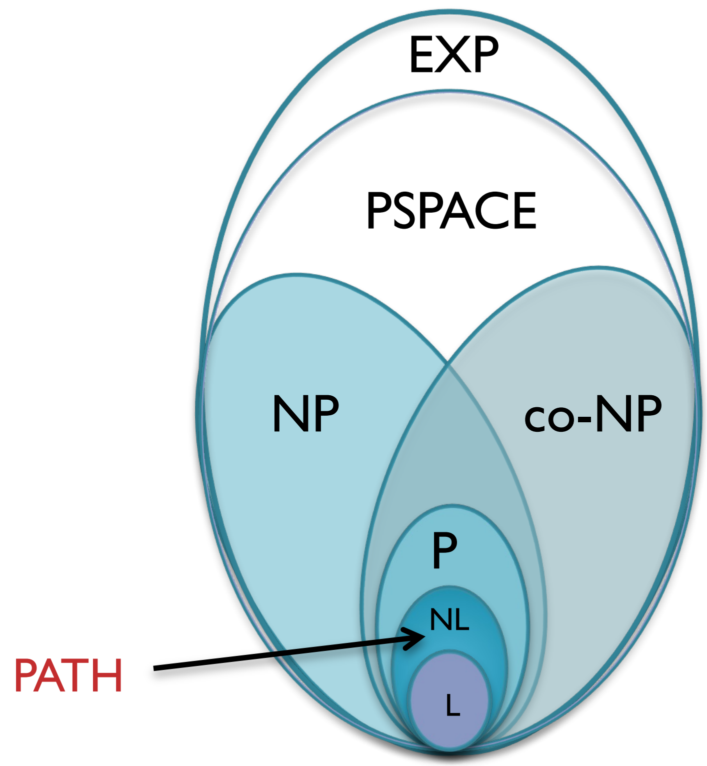
$$L = DSPACE(\log n)$$
$$NL = NSPACE(\log n)$$
$$PSPACE = \bigcup_{c > 0} DSPACE(n^c)$$

- Theorem. $L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP$.

- Are there **natural problems** in L, NL and PSPACE ?

PATH: A canonical problem in NL

- PATH = {(G,s,t) : G is a directed graph having a path from s to t}.
- Obs. PATH is in NL.



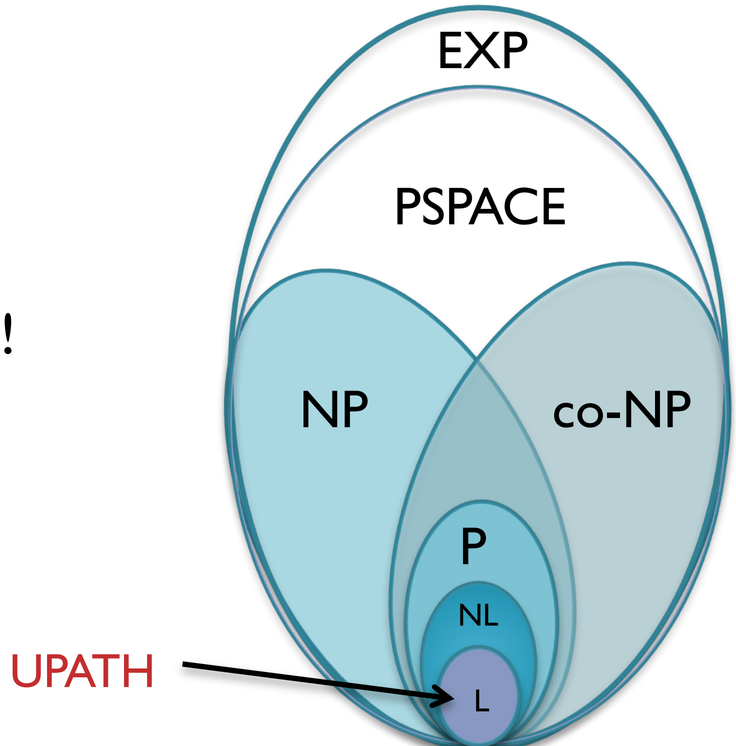
UPATH: A problem in L

- UPATH = {(G,s,t) : G is an undirected graph having a path from s to t}.
- Theorem *(Reingold 2005)*. UPATH is in L.

Is PATH in L ?

If yes, then L = NL !

(will prove later)



Recap: Space Hierarchy Theorem

- Theorem. *(Stearns, Hartmanis & Lewis 1965)* If f and g are space-constructible functions and $f(n) = o(g(n))$, then $\mathrm{SPACE}(f(n)) \subsetneq \mathrm{SPACE}(g(n))$.

- Proof. Homework.

- Theorem. $\mathrm{L} \subsetneq \mathrm{PSPACE}$.

Recap: Savitch's theorem

- Theorem. NSPACE(S(n)) $\subseteq$ DSPACE(S(n)^2), where S(n) is space constructible. (So, PSPACE = NPSPACE)
- Proof.
- REACH(C_1, C_2, i) {

  If i = 0 check if C_1 and C_2 are adjacent.

  Else, <u>for every</u> configurations C,

  a_1 = REACH(C_1, C, i-1)

  a_2 = REACH(C, C_2, i-1)

  if a_1=1 & a_2=1, return 1. Else return 0.

  }

Recap: Savitch's theorem

- Theorem. $NSPACE(S(n)) \subseteq DSPACE(S(n)^2)$, where $S(n)$ is space constructible. (So, PSPACE = NPSPACE)

- Proof.

$$Space(i) = Space(i-1) + O(S(n))$$

- Space complexity: $O(S(n)^2)$

$$Time(i) = 2^m.2.Time(i-1) + O(S(n))$$

- Time complexity: $2^{O(S(n)^2)}$

Recall, $NSPACE(S(n)) \subseteq DTIME(2^{O(S(n))})$. There's an algorithm with time complexity $2^{O(S(n))}$, but higher space requirement.

PSPACE-completeness

PSPACE-completeness

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is P = PSPACE ?

PSPACE-completeness

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is P = PSPACE ? …use poly-time Karp reduction!

- Definition. A language L' is *PSPACE-hard* if for every L in PSPACE, $L \leq_p L'$. Further, if L' is in PSPACE then L' is *PSPACE-complete*.

A PSPACE-complete problem

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is P = PSPACE ? …use poly-time Karp reduction!

- Example. $L' = \{(M,w,1^m) : M$ accepts w using m space$\}$

Natural PSPACE-complete problem

- Definition. A *quantified Boolean formula (QBF)* is a formula of the form

$$Q_1 x_1 \; Q_2 x_2 \; \ldots \; Q_n x_n \;\; \varphi(x_1, x_2, \ldots, x_n)$$

Quantifiers $\exists$ or $\forall$

Just a formula on Boolean variables

- A QBF is either *true* or *false* as all variables are quantified. This is unlike a formula we've seen before where variables were unquantified/free.

Natural PSPACE-complete problem

- Example. $\exists x_1 \exists x_2 \ldots \exists x_n \ \varphi(x_1, x_2, \ldots, x_n)$

- The above QBF is true iff φ is satisfiable.

- We could have defined SAT as

  SAT = $\{\exists \mathbf{x}\ \varphi(\mathbf{x}) : \varphi$ is a CNF and $\exists \mathbf{x}\ \varphi(\mathbf{x})$ is true$\}$

  instead of

  SAT = $\{\varphi(\mathbf{x}) : \varphi$ is a CNF and φ is satisfiable$\}$

Natural PSPACE-complete problem

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Quantifiers $\exists$ or $\forall$

Just a formula on Boolean variables

- Homework: By using auxiliary variables (as in the proof of Cook-Levin) and introducing some more $\exists$ quantifiers at the end, we can assume w.l.o.g. that φ is a 3CNF.

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: Easy to see that TQBF is in PSPACE – just think of a suitable **recursive procedure**. We'll now show that every $L \in$ PSPACE reduces to TQBF via poly-time Karp reduction…

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) Let M be a TM deciding L using S(n) = poly(n) space. We intend to come up with a poly-time reduction f s.t.

$$x \in L \overset{f}{\longleftrightarrow} \psi_x \text{ is a true QBF}$$

Size of ψ_x must be bounded by poly(n), where $|x| = n$

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$$x \in L \overset{f}{\longleftrightarrow} \psi_x \text{ is a true QBF}$$

Idea: Form ψ_x in such a way that ψ_x is true iff there's a path from C_{start} to C_{accept} in $G_{M,x}$.

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) f computes S(n) from n (recall, any poly function S(n) is time constructible). It also computes m = O(S(n)), the no. of bits required to represent a configuration in $G_{M,x}$.

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The variables corresponding to the bits of C_1 and C_2 are unquantified/free variables of Δ_i

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) QBF $\Delta_i(C_1,C_2)$ is formed, recursively, as follows:

(first attempt)

$$\Delta_i(C_1,C_2) = \exists C \ \left(\Delta_{i-1}(C_1,C) \wedge \Delta_{i-1}(C,C_2)\right)$$

Issue: Size of Δ_i is **twice** the size of Δ_{i-1} !!

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) QBF $\Delta_i(C_1,C_2)$ is formed, recursively, as follows:

(careful attempt)

$$\Delta_i(C_1,C_2) = \exists C\ \forall D_1 \forall D_2$$

$$\Big(\big((D_1 = C_1 \wedge D_2 = C) \vee (D_1 = C \wedge D_2 = C_2)\big) \Rightarrow \Delta_{i-1}(D_1,D_2) \Big)$$

Natural PSPACE-complete problem

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- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) QBF $\Delta_i(C_1,C_2)$ is formed, recursively, as follows:

(careful attempt)

$$\Delta_i(C_1,C_2) = \exists C\ \forall D_1 \forall D_2$$

$$\Big(\neg\big((D_1 = C_1 \wedge D_2 = C) \vee (D_1 = C \wedge D_2 = C_2)\big) \quad \vee \quad \Delta_{i-1}(D_1,D_2)\ \Big)$$

Note: Size of Δ_i = $O(S(n))$ + Size of Δ_{i-1}

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) Finally,

$$\psi_x = \Delta_m(C_{start}, C_{accept})$$

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- Proof: (contd.) Finally,

$$\psi_x = \Delta_m(C_{start}, C_{accept})$$

- But, we need to specify how to form $\Delta_0(C_1,C_2)$.
- Size of $\psi_x = O(S(n)^2) +$ Size of Δ_0

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- But, we need to specify how to form $\Delta_0(C_1,C_2)$.
- Size of $\psi_x = O(S(n)^2)$ + Size of Δ_0

Remark: We can easily bring all the quantifiers at the beginning in ψ_x (as in a *prenex normal form*).

Natural PSPACE-complete problem

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- Proof: (contd.) Finally,

$$\psi_x = \Delta_m(C_{start}, C_{accept})$$

- But, we need to specify how to form $\Delta_0(C_1, C_2)$.
- Size of $\psi_x = O(S(n)^2)$ + Size of Δ_0 → ??

Adjacent configurations

- Claim. There's an $O(S(n)^2)$-size circuit $\varphi_{M,x}$ on $O(S(n))$ inputs such that for every inputs C_1 and C_2, $\varphi_{M,x}(C_1, C_2) = 1$ iff C_1 and C_2 encode two neighboring configurations in $G_{M,x}$.

- Proof. Think of a linear time algorithm that has the knowledge of M and x, and on input C_1 and C_2 it checks if C_2 is a neighbor of C_1 in $G_{M,x}$.

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- Proof. Think of a linear time algorithm that has the knowledge of M and x, and on input C_1 and C_2 it checks if C_2 is a neighbor of C_1 in $G_{M,x}$. Applying ideas from the proof of Cook-Levin theorem, we get our desired $\varphi_{M,x}$ of size $O(S(n)^2)$.

Size of Δ_0

- Obs. We can convert the circuit $\varphi_{M,x}(C_1, C_2)$ to a quantified CNF $\Delta_0(C_1,C_2)$ by introducing auxiliary variables (as in the proof of Cook-Levin theorem).

- Hence, size of $\Delta_0(C_1,C_2)$ is $O(S(n)^2)$.
- Therefore, size of $\psi_x = O(S(n)^2)$.

Other PSPACE complete problems

- Checking if a player has a winning strategy in certain two-player games, like (generalized) Hex, Reversi, Geography etc.
- Integer circuit evaluation (*Yang 2000*).
- Implicit graph reachability.
- Check the wiki page: *https://en.wikipedia.org/wiki/List_of_PSPACE-complete_problems*

Log-space reductions

NL-completeness

- Recall again, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is L = NL ?

NL-completeness

- Recall again, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is L = NL ? …poly-time (Karp) reductions are much too powerful for L.
- We need to define a suitable *'log-space'* reduction.

Log-space reductions

$x \xrightarrow{\text{Log-space TM}} f(x)$

- Issue: A log-space TM may not have enough space to write down the whole output f(x) in one shot.

…unless we restrict $|f(x)| = O(\log |x|)$, in which case we're severely restricting the power of the reduction.

Log-space reductions

$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$

- Issue: A log-space TM may not have enough space to write down the whole output f(x) in one shot.
- Solution: Have the log-space TM output a bit of f(x).

Log-space reductions

$$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$$

- Issue: A log-space TM may not have enough space to write down the whole output $f(x)$ in one shot.
- Solution: Have the log-space TM output a bit of $f(x)$.

- Definition: A function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ is *implicitly log-space computable* if
  1. $|f(x)| \leq |x|^c$ for some constant c,
  2. The following two languages are in $\mathbf{L}$:

$$L_f = \{(x, i) : f(x)_i = 1\} \quad \text{and} \quad L'_f = \{(x, i) : i \leq |f(x)|\}$$

Log-space reductions

$$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$$

- **Issue:** A log-space TM may not have enough space to write down the whole output $f(x)$ in one shot.
- **Solution:** Have the log-space TM output a bit of $f(x)$.

- **Definition:** A language L_1 is *log-space reducible* to a language L_2, denoted $L_1 \leq_l L_2$, if there's an implicitly log-space computable function f such that

$$x \in L_1 \iff f(x) \in L_2$$

Log-space reductions

$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$

- Issue: A log-space TM may not have enough space to write down the whole output $f(x)$ in one shot.
- Solution: Have the log-space TM output a bit of $f(x)$.

- Claim: If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- Proof: Let f be the reduction from L_1 to L_2, and g the reduction from L_2 to L_3. We'll show that the function $h(x) = g(f(x))$ is implicitly log-space computable which will suffice as,

$$x \in L_1 \iff f(x) \in L_2 \iff g(f(x)) \in L_3$$

Log-space reductions

$$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$$

- **Issue:** A log-space TM may not have enough space to write down the whole output $f(x)$ in one shot.
- **Solution:** Have the log-space TM output a bit of $f(x)$.

- **Claim:** If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- **Proof:** …Think of the following log-space TM that computes $h(x)_i$ from (x, i). Let
  - M_f be the log-space TM that computes $f(x)_j$ from (x, j),
  - M_g be the log-space TM that computes $g(y)_i$ from (y, i).

Log-space reductions

$$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$$

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- Solution: Have the log-space TM output a bit of $f(x)$.

- Claim: If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- Proof: …On input x, simulate M_g on $(f(x), i)$ <u>pretending that</u> $f(x)$ is there in some fictitious tape. During the simulation whenever M_g tries to read a j-th bit of $f(x)$, <u>postpone</u> M_g's computation and start simulating M_f on input (x, j).

Log-space reductions

(x, i) —Log-space TM→ $f(x)_i$

- Issue: A log-space TM may not have enough space to write down the whole output $f(x)$ in one shot.
- Solution: Have the log-space TM output a bit of $f(x)$.

stores M_g's current configuration

- Claim: If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- Proof: …On input x, simulate M_g on $(f(x), i)$ <u>pretending that</u> $f(x)$ is there in some fictitious tape. During the simulation whenever M_g tries to read a j-th bit of $f(x)$, <u>postpone</u> M_g's computation and start simulating M_f on input (x, j). Space usage $= O(\log |f(x)|) + O(\log |x|)$.

Log-space reductions

$$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$$

- Issue: A log-space TM may not have enough space to write down the whole output $f(x)$ in one shot.
- Solution: Have the log-space TM output a bit of $f(x)$.

- Claim: If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- Proof: …On input x, simulate M_g on $(f(x), i)$ pretending that $f(x)$ is there in some fictitious tape. During the simulation whenever M_g tries to read a j-th bit of $f(x)$, postpone M_g's computation and start simulating M_f on input (x, j). Space usage $= O(\log |x|)$.

Log-space reductions

$$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$$

- **Issue:** A log-space TM may not have enough space to write down the whole output $f(x)$ in one shot.
- **Solution:** Have the log-space TM output a bit of $f(x)$.

- **Claim:** If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- **Proof:** …On input x, simulate M_g on $(f(x), i)$ pretending that $f(x)$ is there in some fictitious tape. During the simulation whenever M_g tries to read a j-th bit of $f(x)$, postpone M_g's computation and start simulating M_f on input (x, j). This shows L_h is in L.

Log-space reductions

$$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$$

- Issue: A log-space TM may not have enough space to write down the whole output $f(x)$ in one shot.
- Solution: Have the log-space TM output a bit of $f(x)$.

- Claim: If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- Proof: …Similarly, L'_h is in L and so h is implicitly log-space computable.

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- Claim: If $L_1 \leq_l L_2$ and $L_2 \in \mathbf{L}$ then $L_1 \in \mathbf{L}$.
- Proof: Same ideas. (*Homework*)

NL-completeness

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  PATH = {(G,s,t) : G is a digraph having a path from s to t}.

- Theorem: PATH is NL-complete.
- Proof: We've already shown that $PATH \in NL$. Now we'll show that for every $L \in NL$, $L \leq_l PATH$. We need to come up with an implicitly log-space computable function f s.t.

$$x \in L \iff f(x) \in PATH$$

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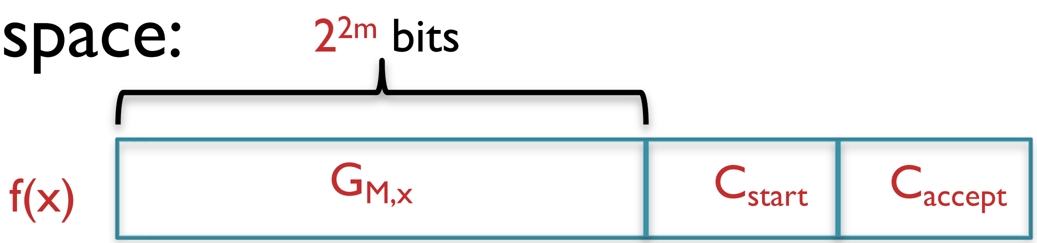

If $i > 2^{2m}$ then i indexes a bit in the (C_{start}, C_{accept}) part of f(x); so $f(x)_i$ can be computed by simply writing down C_{start} and C_{accept}.

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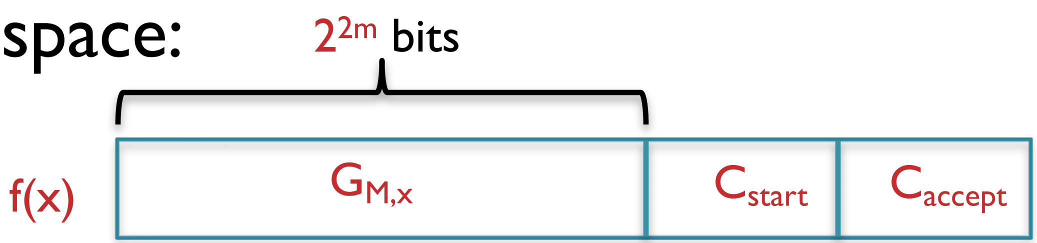

If $i \leq 2^{2m}$ then write i as (C_1, C_2), where C_1 and C_2 are m bits each, and check if C_2 is a neighbor of C_1 in $G_{M,x}$. This takes $O(m)$ space.

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- Theorem: PATH is NL-complete.
- Proof: (contd.) Thus, we've argued that $|f(x)|$ has poly($|x|$) length and $L_f \in L$. Similarly, $L'_f \in L$. So, f defines a log-space reduction from L to PATH.

Other NL-complete problems

- Reachability in directed acyclic graphs.
- Checking if a directed graph is strongly connected.
- 2SAT.
- Determining if a word is accepted by a NFA.