Topics in complexity theory: Expander graphs - Assignment 1

Due date: Feb 12, 2014

General instructions:

- Write your solutions by furnishing all relevant details. (You may assume the results whose proofs are already worked out in the class).
- You are strongly urged to solve the problems by yourself.
- If you discuss with someone else or refer to any material (other than the class notes) then please put a reference in your answer script stating clearly whom or what you have consulted with and how it has benifited you. We would appreciate your honesty.
- The following exercise problems are taken from Salil Vadhan's survey article on *Psuedo-randomness.* If you need any clarification, please ask the instructors.

Total: 50 points

In the following problems, $\mathbb{R}$ is the set of real numbers and $\mathbb{N}$ is the set of natural numbers.

1. (**18 points**) (*Spectral Graph theory*) Let M be the random-walk matrix for a d-regular undirected graph $G = (V, E)$ on n vertices. We allow G to have self-loops and multiple edges. Recall that the uniform distribution $u = (1/n, \ldots, 1/n)$ is an eigenvector of M of eigenvalue $\lambda_1 = 1$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the eigenvalues of M. Prove the following statements.

   (a) (**2 points**) All eigenvalues of M have absolute value at most 1.

   (b) (**3 points**) G is connected if and only if 1 is an eigenvalue of M of multiplicity at least 2.

   (c) (**3 points**) Suppose G is connected. Then G is bipartite if and only if -1 is an eigenvalue of M.

   (d) (**5 points**) If G is connected then all eigenvalues of M other than λ_1 are at most $1 - 1/\mathsf{poly}(n, d)$. To do this, it may help to first show that λ_2 equals

   $$\max_x \langle xM, x \rangle = 1 - \frac{1}{d} \cdot \min_x \sum_{\{i,j\} \in E} (x_i - x_j)^2,$$

   where the maximum/minimum is taken over all vectors $x = (x_1, \ldots, x_n)$ of length 1 such that $\sum_i x_i = 0$, and $\langle x, y \rangle = \sum_i x_i y_i$ is the standard inner product.

(e) (**5 points**) If G is connected and nonbipartite then all eigenvalues of M (other than 1) have absolute value at most $1 - 1/\mathsf{poly}(n,d)$.

2. (**10 points**) (*Hitting time and eigenvalues for directed graphs*) For a digraph $G = (V,E)$, define its *hitting time* as

$$\mathrm{hit}(G) = \max_{i,j \in V} \min\{t : \Pr[\text{a random walk of length } t \text{ started at } i \text{ visits } j] \geq 1/2\}.$$

For a *regular* digraph G with random-walk matrix M, define

$$\lambda(G) := \max_{\pi} \frac{\|\pi M - u\|}{\|\pi - u\|} = \max_{x \perp u} \frac{\|xM\|}{\|x\|},$$

where the first maximization is over all *probability distribution* $\pi \in [0,1]^n$ and the second is over all vectors $x \in \mathbb{R}^n$ such that $x \perp u$ (as usual, u is the uniform distribution).

(a) (**2 points**) Show that for every n, there exists a digraph G with n vertices, outdegree 2, and $\mathrm{hit}(G) = 2^{\Omega(n)}$.

(b) (**5 points**) Let G be a *regular* digraph with random-walk matrix M. Show that $\lambda(G) = \sqrt{\lambda(G')}$, where G' is the undirected graph whose random-walk matrix is MM^T, and $\lambda(G)$.

(c) (**3 points**) A digraph G is called *Eulerian* if it is connected and every vertex has the same indegree as outdegree. (For example, if we take a connected undirected graph and replace each undirected edge $\{u,v\}$ with two directed edges (u,v) and (v,u), we obtain an Eulerian digraph. These are precisely the digraphs that have Eulerian circuits, i.e. closed paths that visit all vertices and use every directed edge exactly once.) Show that if G is an n-vertex Eulerian digraph of maximum degree d, then $\mathrm{hit}(G) = \mathsf{poly}(n,d)$.

3. (**8 points**) (*Error reduction for free*) Show that if a problem has a BPP algorithm with constant error probability, then it has a BPP algorithm with error probability $1/n$ that uses *exactly* the same number of random bits.

4. (**14 points**) (*Limits on spectral expansion*) Let G be a d-regular n-vertex undirected graph and T_d be the infinite d-regular tree. For a graph H and $\ell \in \mathbb{N}$, let $p_\ell(H)$ denote the probability that if we choose a random vertex v in H and do a random walk of length 2ℓ, we end back at vertex v.

(a) (**6 points**) Show that $p_\ell(G) \geq p_\ell(T_d) \geq C_\ell \cdot (d-1)^\ell / d^{2\ell}$, where C_ℓ is the ℓ-th Catalan number, which equals the number of properly paranthesized strings in $\{(,)\}^{2\ell}$ — strings where no prefix has more)'s than ('s.

(b) (**5 points**) Show that $n \cdot p_\ell(G) \leq 1 + (n-1) \cdot \lambda(G)^{2\ell}$, where $\lambda(G)$ is the absolute value of the second largest eigenvalue (in magnitude) of the random-walk matrix of G.

(c) (**3 points**) Using the fact that $C_\ell = \binom{2\ell}{\ell}/(\ell+1)$, prove that

$$\lambda(G) \geq \frac{2\sqrt{d-1}}{d} - o(1),$$

where the $o(1)$ term vanishes as $n \to \infty$ (and d is held constant).