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Covering a set of points in a plane using two parallel rectangles ☆

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ABSTRACT

In this paper we consider the problem of finding two parallel rectangles in arbitrary orientation for covering a given set of n points in a plane, such that the area of the larger rectangle is minimized. We propose an algorithm that solves the problem in $O(n^3)$ time using $O(n^2)$ space. Without altering the complexity, our approach can be used to solve another optimization problem namely, minimize the sum of the areas of two arbitrarily oriented parallel rectangles covering a given set of points in a plane.

1. Introduction

Two rectangles are said to be parallel when each side of a rectangle is parallel to a side of the other rectangle. In this paper we consider the following problem **P**: Given a set S of n points in a plane, locate two parallel rectangles D_1 and D_2 that cover S, such that the area of the larger rectangle is minimized, among all possible covers by the two parallel rectangles. Fig. 1 depicts a possible location of the two parallel rectangles covering point set S in a plane.

Bespamyatnikh and Segal [1] considered a similar problem in d-dimensional space, but using axis-parallel boxes to cover the points, and obtained a time complexity of $O(n \log n + n^{d-1})$. For the problem of covering points by a pair of parallel rectangles in arbitrary orientation, Jaromczyk and Kowaluk [2] devised an $O(n^2)$ time algorithm that decides whether two parallel rectangles with side lengths a, b and c, d respectively can cover the given point set. Jaromczyk and Kowaluk [2] also proposed an $O(n^2)$ time algorithm that finds a covering of S using two parallel squares in arbitrary orientation that optimizes the sizes of the squares with respect to their side lengths. This was followed by a work by Katz et al. [3] that solved the problem of finding two parallel squares in arbitrary orientation that covers S, with the additional constraint that the center points of squares belong to S, such that the area of the

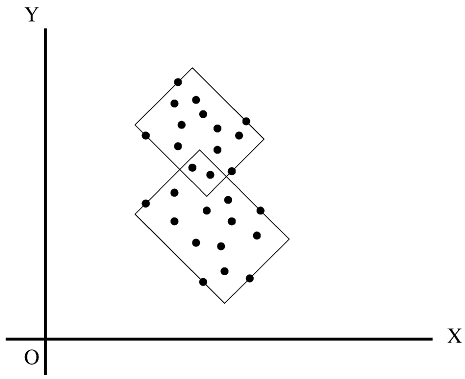

Fig. 1. Covering by two parallel rectangles.

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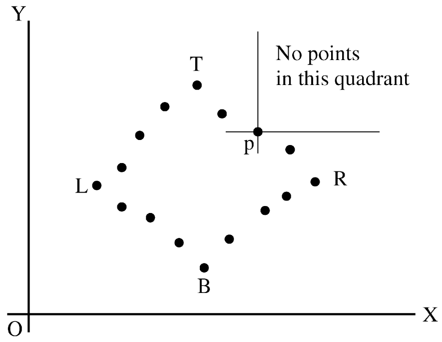

Fig. 2. Points in $\wp(0)$.

larger square is minimized. The running time of their algorithm is $O(n^2 \log^4 n)$ and uses $O(n^2)$ space. In the same paper they also presented an $O(n^3 \log^2 n)$ time algorithm using $O(n^2)$ space to locate the optimum squares where each square is allowed to rotate independently (that is, the squares need not be parallel). In this paper, we propose an algorithm that solves problem **P** in $O(n^3)$ time using $O(n^2)$ space. This result compares well with [2] and [3] because a rectangle has an added degree of freedom compared to that of a square.

The problem finds application in VLSI physical design for accommodating specified locations like hot spots, power pins into two parallel rectangles. It also has application in the field of geographical information systems for map data modeling, map overlay, map labeling, etc. Moreover, there are applications in image processing and facility location problems.

2. Preliminaries

Let $S = \{p_1, p_2, \ldots, p_n\}$ be the set of n points. $x(p)$ and $y(p)$ represent the x- and y-coordinates of point p respectively. $XY(\alpha, p)$ denotes the coordinate system obtained by making a counter-clockwise rotation of the original axis by an angle α, $\alpha \in [0, \pi/2)$, around origin O and by shifting the origin from O to point p using linear translation only. $\wp(\alpha)$ for $\alpha \in [0, \pi/2)$ denotes a subset of S such that $p \in \wp(\alpha)$ if and only if there exists at least one closed quadrant of the system $XY(\alpha, p)$ that has no point in $S\backslash\{p\}$ (see Fig. 2). Note that the set *α-silhouette*(S) defined by Jaromczyk and Kowaluk [2] is same with the set $\wp(\alpha)$.

Let D_1 and D_2 be two optimal parallel rectangles and the sets of points covered by rectangles D_1 and D_2 are denoted by S_1 and S_2 respectively, where $S_1 \cup S_2 = S$. Surely D_1 and D_2 are smallest axis-aligned rectangles, covering points in S_1 and S_2 respectively, with respect to some system $XY(\alpha, O)$. Let $D(\alpha)$ be the smallest axis-aligned rectangle in that coordinate system that encloses all points of S. $T(\alpha)$ is an ordered set containing the top, left, bottom and right boundary points of $D(\alpha)$ in that order. The set $T(\alpha)$ will be called the *extreme points* of S in the system $XY(\alpha, O)$. The notation XY represents the coordinate system $XY(\alpha, p)$ at α equal to zero and p at origin O. For simplicity we shall assume that no two points in S have the same x- or y-coordinate in the original coordinate system XY. We shall also assume that no three points are collinear and no three points form an angle of $\frac{\pi}{2}$. These cases can be tackled in a similar way with some minor modifications to our approach.

Now consider the optimum rectangles D_1 and D_2 are parallel to the axes of XY (also known as *isothetic* rectangles). We have the following results.

Lemma 1. *If the rectangles D_1 and D_2 overlap, then each side of the rectangles must contain a point from set $\wp(0)$. Moreover, given the sorted sequences of S on the basis of their x- and y-coordinates, $\wp(0)$ can be computed in $O(n)$ time.*

Proof. The first statement follows from the optimal nature of D_1 and D_2. The set $\wp(0)$ can be computed in $O(n)$ time using a sweep line algorithm. □

Glozman et al. [4] pointed out an $O(n \log n)$ time algorithm for evaluating two optimum size axis parallel rectangles for covering the point set S. Later Bespamyatnikh and Segal [1] proposed a theorem that basically concludes the following results.

Result 1. *(See [1].) Given a set S of n points in a plane along with two sorted sequences of those points on the basis of their x- and y-coordinates, the problem of locating two axis-parallel rectangles that cover S, so as to minimize the area of the larger rectangle, can be solved in $O(n)$ time.*

3. Covering by orientation independent parallel rectangles

Here we consider the case where two parallel rectangles cover the point set S and they may be placed in any orientation and form an angle in range $[0, 2\pi]$ with x-axis in XY coordinate system. Note that those pair of optimal rectangles may or may not overlap. We consider these two cases separately. With the intent of pruning our search space, we explore some characterization of the optimal rectangles.

Let $CH(S)$ be the convex hull of point set S and $\mathcal{A}(S)$ be the set of angles formed by the edges of $CH(S)$ with the x-axis. For the problem of covering a set of n points using a single rectangle in any arbitrary orientation having minimum area, Freeman and Shapira [5] suggested an interesting characterization as stated below.

Result 2. *(See [5].) Let $D(\alpha)$ be the minimum area rectangle enclosing points of S with one of its sides making an angle $\alpha \in [0, \frac{\pi}{2})$ with the x-axis of the XY coordinate system. If the area of $D(\alpha)$ achieves a local optimal value then either α or $\alpha + \frac{\pi}{2}$ must coincide with one of the elements in set $\mathcal{A}(S)$.*

Suppose D_1 and D_2 be two optimal parallel rectangles as described in Section 2.

Lemma 2. *Let one of the sides of rectangles D_1 and D_2 be inclined at angle α to the x-axis in XY plane. Then at least one of the following two conditions is true*:

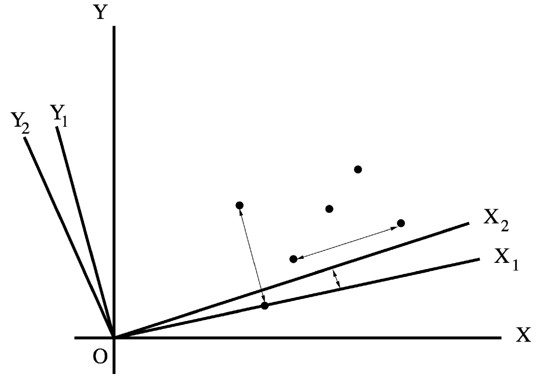

Fig. 3. $X_1 O X_2$ is a separable interval.

(i) *there exists at least a pair of points p, q in S such that* $x(p) = x(q)$ *or* $y(p) = y(q)$ *with respect to the coordinate system* $XY(\alpha, O)$;
(ii) $Area(D_1) = Area(D_2)$.

Proof. Suppose that neither condition (i) nor condition (ii) hold. Let S_1 and S_2 be the subsets of points in S that are covered by D_1 and D_2 with $Area(D_1) > Area(D_2)$. Since condition (i) is not satisfied, there exist infinitesimally small rotations of the two rectangles, in both clockwise and counter-clockwise directions such that the rectangles still cover the same subsets S_1 and S_2, while their boundary points remain unaltered. From Result 2 it follows that there exists a small angle of rotation $\delta\alpha$ of the system $XY(\alpha, O)$ to $XY(\alpha + \delta\alpha, O)$ such that the area of rectangle D_1 is reduced while D_1 remains the larger of the two rectangles. □

The coordinate system $XY(\alpha, O)$ for some α ($\in [0, 2\pi]$) is termed as *separable* for S if no two points p, q in S have same x- or y-coordinates in that system. An open interval (a, b) within the range $[0, 2\pi]$ is called a *separable interval* for S if for any $\alpha \in (a, b)$, the system $XY(\alpha, O)$ is separable for S (see Fig. 3).

Let $\chi(\alpha) = \langle \chi_1(\alpha), \chi_2(\alpha), \ldots, \chi_n(\alpha) \rangle$ and $\psi(\alpha) = \langle \psi_1(\alpha), \psi_2(\alpha), \ldots, \psi_n(\alpha) \rangle$ be two sequences representing the points in S sorted with respect to their x- and y-coordinates respectively in the coordinate system $XY(\alpha, O)$.

Observation 1. *As long as α belongs to the same separable interval, Δ, the two sequences $\chi(\alpha)$ and $\psi(\alpha)$ remain unaltered.*

Let $D(\alpha, S')$ denote the smallest axis aligned rectangle in coordinate system $XY(\alpha, O)$ enclosing the points in set S', $S' \subseteq S$. Points on the boundary of the rectangle $D(\alpha, S')$ will be termed as the *extreme points* of $D(\alpha, S')$.

Observation 2. *Consider any separable interval Δ and a subset S', $S' \subseteq S$. Rectangles $D(\alpha, S')$ for all $\alpha \in \Delta$ have the same set of extreme points.*

Let S_1 and S_2 be subsets of points in S with $S_1 \cup S_2 = S$ (S_1 and S_2 need not be disjoint). Λ_1 and Λ_2 be sets containing the extreme points of the minimum enclosing isothetic rectangles covering set S_1 and S_2 respectively in some coordinate system $XY(\alpha, O)$ for some $\alpha \in \Delta$. From Observation 2 we can conclude that the sets Λ_1, Λ_2 remain unaltered for all $\alpha \in \Delta$.

Lemma 3. *Given a separable interval Δ, the sets S_1, S_2 and the sets Λ_1, Λ_2, if there exist an angle $\alpha \in \Delta$ such that $Area(D(\alpha, S_1)) = Area(D(\alpha, S_2))$ then α can be computed in constant time. There can be at most two different solutions for α in any separable interval Δ.*

Proof. Let $\Delta : (\phi_i, \phi_f)$ be a *separable interval* for the set S. The areas of rectangles $D(\alpha, S_1)$ and $D(\alpha, S_2)$ can be expressed as a trigonometric function of α. Equating $Area(D(\alpha, S_1))$ and $Area(D(\alpha, S_2))$ we get an equation of the form, $F(\alpha) = C$, where $F(\alpha)$ is a sinusoidal function and C is a constant. As α varies only in the range $[0, \frac{\pi}{2}]$, the equation $F(\alpha) = C$ can have at most two solutions. □

Suppose Θ be the set of angles within $[0, \pi]$, formed by a line joining a pair of points in S, with the x-axis of the XY coordinate system. Notice that, for any $\theta \in \Theta$, $0 \leqslant \theta \leqslant \pi/2$, the coordinate systems $XY(\theta, O)$ and $XY(\pi/2 + \theta, O)$ are not separable for S and for $\pi/2 \leqslant \theta \leqslant \pi$, the coordinate systems $XY(\theta, O)$ and $XY(\theta - \pi/2, O)$ are not separable. Observe that for a non-separable coordinate system $XY(\alpha, O)$ with $0 \leqslant \alpha \leqslant \pi/2$, either α or $\alpha + \pi/2$ must be an element of Θ. Consider the set $\Phi = \{\phi\colon \phi \in [0, \pi/2], \phi \in \Theta$ or $(\phi + \pi/2) \in \Theta\}$. The cardinality of Φ is bounded by $\binom{n}{2}$. Let the elements $\phi_1, \phi_2, \ldots, \phi_k$ of set Φ be in increasing order according to their values and this ordered sequence is referred by Φ itself. From our assumption, as no two points have same x- or y-coordinates in XY, $\phi_1 > 0$. We introduce a new element ϕ_0 as zero and therefore the set of intervals $\Delta = \{\Delta_1, \Delta_2, \ldots, \Delta_k\}$ where $\Delta_i = (\phi_{i-1}, \phi_i)$ for $i = 1, 2, \ldots, k$ are separable intervals in range $[0, \pi/2]$. We introduce one more separable interval $\Delta_{k+1} = (\phi_k, \pi/2]$ and hence $\Delta = \{\Delta_1, \Delta_2, \ldots, \Delta_k, \Delta_{k+1}\}$ is the exhaustive set of separable intervals in $[0, \pi/2]$. Suppose p and q of S have same x- or y-coordinate values in the coordinate system $XY(\phi_i, O)$ ($\phi_i \in \Phi$). Given the sequences $\chi(\phi_i)$ and $\psi(\phi_i)$, if the indices of p and q on those sequences are available, then for any α in separable interval Δ_i or Δ_{i+1} the sequences $\chi(\alpha)$ and $\psi(\alpha)$ can be evaluated, in constant time, by modifying $\chi(\phi_i)$ and $\psi(\phi_i)$ using constant number of interchanges in the lists.

3.1. The rectangles do not overlap

In this section, we present an efficient algorithm to locate two parallel orientation independent disjoint optimal rectangles covering the point set S.

Observation 3. *Suppose that the sets A_j and A'_j contain the first j ($1 \leqslant j \leqslant n$) elements of sequence $\chi(\alpha)$ and the remaining elements of $\chi(\alpha)$ respectively, for some $\alpha \in \Delta_i$. Then the extreme points of the minimum enclosing isothetic rectangles of sets A_j, A'_j in system $XY(\beta, O)$ are same for all $\beta \in \Delta_i$.*

Moreover, the extreme points of the minimum enclosing isothetic rectangles of set A_j, A'_j for all values of j, $1 \leqslant j \leqslant n$, can be evaluated in $O(n)$ time.

Optimal_disjoint_rectangles(S)

1. Compute Θ and enumerate the set Φ. With each element $\phi \in \Phi$ associate the points of S that have the same x- or y-coordinates in system $XY(\phi, O)$.
2. Sort the elements of Φ according to their angular values.
3. Generate $\chi(\phi_0)$ and $\psi(\phi_0)$ by sorting the points in S with respect to the system XY. Fix two pointers g and h with each point p of S such that for $g(p) = l$ implies $\chi_l(\phi_0) = p$ and similarly $h(p) = l$ implies $\psi_l(\phi_0) = p$.
4. **for** $i = 1$ to $k+1$ **do**
   (a) Consider coordinate system $XY(\alpha, O)$ for any $\alpha \in \Delta_i$. Generate $\chi(\alpha)$ and $\psi(\alpha)$ from $\chi(\phi_{i-1})$ and $\psi(\phi_{i-1})$. Using the techniques as described in Section 2, Lemma 3 and Observation 3 find two optimum size parallel non-overlapping enclosing rectangles creating angles with x-axis within range $[\phi_{i-1}, \phi_i)$.
   (b) Store the optimal pair of rectangles in the list L.
   endfor
5. Traverse the list L to find the optimum placement.

3.1.1. Analysis of the algorithm

In Step 1, the set Φ is generated by considering all lines joining pairs of points of set S and sorting the elements of Φ to generate the sequence Φ in $O(n^2 \log n)$ time. Formation of sets $\chi(\phi_0)$ and $\psi(\phi_0)$ and the maintenance of pointers g and h can be done within the same complexity. Note that, updating of sequences $\chi(\alpha)$ and $\psi(\alpha)$ for changing the separable interval from Δ_i to Δ_{i+1} using pointer g and h can be done in constant time. We can locate the optimum pair of parallel non-overlapping enclosing rectangles in the coordinate system $XY(\phi_i, O)$ in $O(n)$ time. By Lemma 3 we can also find the optimum placement of a given pair of parallel non-overlapping enclosing rectangles, within separable interval Δ_i, in $O(1)$ time. Hence, we have the following theorem.

Theorem 1. *The problem of finding two mutually parallel non-overlapping rectangles, such that the area of the larger rectangle is minimized, can be solved in $O(n^3)$ time.*

Proof. The cardinality of the set Φ is $O(n^2)$. The sequences $\chi(\alpha)$ and $\psi(\alpha)$ for $\alpha \in \Delta_{i+1}$ can be computed by updating the same sequences for the separable interval Δ_i in constant time. Now from Result 1, we can conclude the theorem. □

3.2. The rectangles may overlap

In this section, we present an algorithm to locate two parallel orientation independent optimal rectangles, with nonempty intersection region, that cover the point set S.

From the discussion in Section 2 it follows that, only the points in $\wp(\alpha)$ decide the location of the optimum pair of parallel rectangles with nonempty intersection region in the coordinate system $XY(\alpha, O)$. It is simple to observe that, if a point p belongs to $\wp(\alpha)$ for some α in a separable interval Δ_i then p remains in $\wp(\beta)$ for all $\beta \in \Delta_i$.

Let B_1 and B_2 be the ordered sets of boundary points (ordered as top, left, bottom and right boundary points) that define the two isothetic overlapping rectangles in the system $XY(\alpha, O)$ such that together they cover the point set S and none of the boundary points of one rectangle is covered by the other rectangle. Such a tuple (B_1, B_2) is called a *placement* in the system $XY(\alpha, O)$. Note that the optimum orientation independent overlapping parallel pair of rectangles must be a *placement* in some coordinate system.

Observation 4. *There can be at most $O(n^2)$ distinct placements in any given system $XY(\alpha, O)$.*

Observation 5. *All the distinct placements in a system $XY(\alpha, O)$ remain unaltered for all α belonging to a separable interval Δ_i.*

Observation 6. *Let $S' \subset S$ be the set of points having same x- or y-coordinate values in the coordinate system $XY(\phi_i, O)$. A placement (B_1, B_2) in a separable interval Δ_i is also a placement in the interval Δ_{i+1} (and vice versa), if none of the points from $B_1 \cup B_2$ lies in S'.*

Recall (from Section 2) the definition of *extreme points* of S in a coordinate system $XY(\alpha, O)$, as the ordered set of boundary points $T(\alpha)$ of the smallest axis-aligned rectangle $D(\alpha)$ enclosing the points in S. Surely, the extreme points of S remain unaltered for all $\alpha \in \Delta_i$. Let $T_i = T(\alpha)$ be the extreme points of S in $XY(\alpha, O)$ where $\alpha \in \Delta_i$, $1 \leqslant i \leqslant k+1$.

Lemma 4. *Let p be a point in $\wp(\alpha)$, $\alpha \in \Delta_i$, such that $p \notin T_i$. Then all placements (B_1, B_2) in the interval Δ_i, with $p \in B_1 \cup B_2$, can be identified in $O(n)$ time, provided we know the sequences $\chi(\alpha)$ and $\psi(\alpha)$.*

Proof. Note that, the sequences $\chi(\alpha)$ and $\psi(\alpha)$ can be generated from the corresponding sequences in the previous interval Δ_{i-1}, using constant number of interchanges in the lists $\chi(\Delta_{i-1})$ and $\psi(\Delta_{i-1})$. Since we assume that the two rectangles overlap, a pair of sides of each of the two rectangles $D(\alpha, B_1)$ and $D(\alpha, B_2)$ are fixed by the four *extreme points* of the system $XY(\alpha, O)$. Using the sequences $\chi(\alpha)$ and $\psi(\alpha)$, order the set $\wp(\alpha)$ in such a way that by fixing one *extreme* point of one of the rectangles at p one can use a sweep line algorithm to position the only other remaining extreme point of the rectangle and construct all the $O(n)$ placements (B_1, B_2) in $XY(\alpha, O)$, with $p \in B_1 \cup B_2$, in $O(n)$ time. □

Lemma 5. *If $T_i = T_{i+1}$ then the number of placements in the separable interval Δ_{i+1} that differ from the placements in the separable interval Δ_i is at most $O(n)$ and vice versa. Moreover, all these differing placements can be identified in $O(n)$ time.*

Proof. Although there are $O(n^2)$ *distinct placements* in any given system $XY(\alpha, O)$, it is evident that any *placement*

(B_1, B_2) in the separable interval Δ_{i+1} that differs from all the placements in Δ_i must have p or q belonging to $B_1 \cup B_2$ (see Observation 6). Assume without any loss in generality that all the four points of T_i are distinct. If none of p and q belongs to T_i then there are only $O(n)$ *distinct placements* of the form (B_1, B_2) with p or q belonging to $B_1 \cup B_2$. If $p \in T_i$ and $q \notin T_i$, then each *placement* (B_1, B_2) in the separable interval Δ_{i+1} that differs from all the placements in Δ_i must have $q \in B_1 \cup B_2$. All the *differing placements* in the previous two cases can be identified in $O(n)$ time (by Lemma 4). The only other case is when both p and q belong to T_i. Since $T_i = T_{i+1}$, either p, q are the topmost and bottommost points of $D(\alpha)$ or the leftmost and rightmost points of $D(\alpha)$. Assume without any loss of generality that p, q are the topmost and bottommost points of $D(\alpha)$ respectively. Then the placements (B_1, B_2) that differ between the intervals Δ_{i+1} and Δ_i are exactly those where at least one of p, q is either the left or right extreme point of the rectangle formed by B_1 or B_2. By an argument similar in spirit to that of Lemma 4, it can be shown that there are only $O(n)$ such placements all of which can be identified in $O(n)$ time. □

Theorem 2. *There is a total of $O(n^3)$ distinct placements in the interval $[0, \frac{\pi}{2}]$, all of which can be identified in $O(n^3)$ time.*

Proof. Note that there are $O(n^2)$ separable intervals. To start with, there are $O(n^2)$ distinct placements in the separable interval Δ_1. If $T_i = T_{i+1}$, then in the separable interval Δ_{i+1} an additional of $O(n)$ new placements are introduced (Lemma 5). The set T_i may differ from T_{i+1} for a maximum of h times, where h is the number of edges of the convex hull of S, and in each such cases at most $O(n^2)$ new placements are introduced. Hence the result follows. □

An instance can be generated where the number of possible distinct placements in the interval $[0, \frac{\pi}{2}]$ is $\Omega(n^3)$. Below we describe an algorithm for locating a placement that constitutes of an optimum pair of parallel overlapping rectangles covering point set S.

3.2.1. Data structure

Simple arrays and lists are sufficient to construct our data structures. An array M stores the set of points in S. A point in M (say p) that is in $\wp(\alpha)$ for some α in $[0, \frac{\pi}{2}]$, maintains a list structure namely, *placement list* that keeps all placements of the form (B_1, B_2) such that $p \in B_1 \cup B_2$. Each placement $C = (B_1, B_2)$ in a *placement list* associates with it an interval (β_i, β_f) such that for all $\beta \in (\beta_i, \beta_f)$, (B_1, B_2) is a placement in $XY(\beta, O)$. We shall use the variables *C.Interval.start* and *C.Interval.finish* to indicate β_i and β_f respectively.

Optimal_overlapping_rectangles(S)

1. Compute Θ and generate the sequences Φ, $\chi(\phi_0)$ and $\psi(\phi_0)$ as in Steps 1, 2 and 3 of the algorithm Optimal_disjoint_rectangles.
2. Initialize the array M and form all the *placement lists* with the *placements* in XY. For each placement C, initialize $C.Interval.start = 0$.
3. Find the location of the pair of optimum axis-aligned rectangles with nonempty region of intersection in the system XY. Store this optimum location along with the area of the larger rectangle in a variable $\mathcal{O}$.
4. **for** $i = 1$ to k **do**
   (a) Consider coordinate system $XY(\alpha, O)$ for any $\alpha \in \Delta_{i+1}$. Let p and q be the points with equal x- or y-coordinates in the system $XY(\phi_i, O)$. Generate $\chi(\alpha)$ and $\psi(\alpha)$ from the previous sequences $\chi(\gamma)$ and $\psi(\gamma)$ where $\gamma \in \Delta_i$ and form the set $\wp(\alpha)$.
   (b) **if** $T_i = T_{i+1}$ **then**
   If $p, q \notin T_i$, delete all the placements from the placement lists of points p, q and form the placement lists of p and q, anew, by using set $\wp(\alpha)$, $\alpha \in \Delta_{i+1}$. Else if $p \in T_i, q \notin T_i$, delete all the placements from the placement list of point q and form the placement list, anew. Else, if $p, q \in T_i$, deletion and insertion of the placement lists of p and q are handled based on the arguments given in the proof of Lemma 5.
   else
   Delete all the placements from all the placement lists. Form the array M and all the placement lists, anew, by using the set $\wp(\alpha)$, $\alpha \in \Delta_{i+1}$.
   endif
   (c) For each deleted placement $C = (B_1, B_2)$, assign $C.Interval.finish = \phi_i$. Solve for $Area(D(\beta, B_1)) = Area(D(\beta, B_2))$ where $\beta \in (C.Interval.start, C.Interval.finish)$ and update the optimum location stored in the variable $\mathcal{O}$. Also find the locations of the placement C in the systems $XY(C.Interval.start, O)$ and $XY(C.Interval.finish, O)$, and evaluate the areas of the rectangles $D(\beta, B_1)$ and $D(\beta, B_2)$ at $\beta = C.interval.start$ and $\beta = C.interval.finish$. Update the optimum location stored in $\mathcal{O}$ by comparing it with these two locations.
   (d) For each newly formed placement C, assign $C.Interval.start = \phi_i$.
5. Delete all the placements from all the placement lists. For each deleted placement $C = (B_1, B_2)$, assign $C.Interval.finish = \phi_i$ and solve for $Area(D(\beta, B_1)) = Area(D(\beta, B_2))$ where $\beta \in (C.Interval.start, C.Interval.finish)$. Also find the locations of the placement C in the systems $XY(C.Interval.start, O)$ and $XY(C.Interval.finish, O)$. Update the optimum location in $\mathcal{O}$ accordingly.

3.2.2. Complexity of the algorithm

At each iteration of the for loop at Step 4, if $T_i = T_{i+1}$ then $O(n)$ new placements are added and $O(n)$ placements are deleted. If h is the number of edges of $CH(S)$ then for only h iterations of the for loop $O(n^2)$ placements are added or deleted. Therefore, time complexity of the algorithm is $O(n^3)$. Moreover, from Observation 4 we conclude that the space complexity of the algorithm is $O(n^2)$.

Theorem 3. *The problem of locating two parallel rectangles covering a given set of points S in a plane, such that the area of the* *larger rectangle is minimized, can be solved in $O(n^3)$ time and using $O(n^2)$ space.*

4. Conclusion

Our algorithm provides a simple way to search all the $O(n^3)$ distinct *placements* in the interval $[0, \frac{\pi}{2}]$ in $O(n^3)$ time. Without altering the complexity, this technique may be used to solve related problems of covering like finding a cover by two parallel rectangles such that the sum of area of the rectangles is minimized. In that case the only changes in the algorithm are at Steps 4(c) and 5, where we minimize the function $A(\beta) = \mathit{Area}(D(\beta, B_1)) + \mathit{Area}(D(\beta, B_2)$ by equating the derivative of $A(\beta)$ with respect to β to zero in the interval $\beta \in (C.\mathit{Interval.start}, C.\mathit{Interval.finish})$. We conclude by suggesting that it will be interesting to inspect the complexities of these optimization problems when the rectangles *need not* be parallel, i.e. when they are allowed to move freely, independent of one another.

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