Rapid Mixing in Symmetric MC's

January 31, 2018

MC's with Constrictions

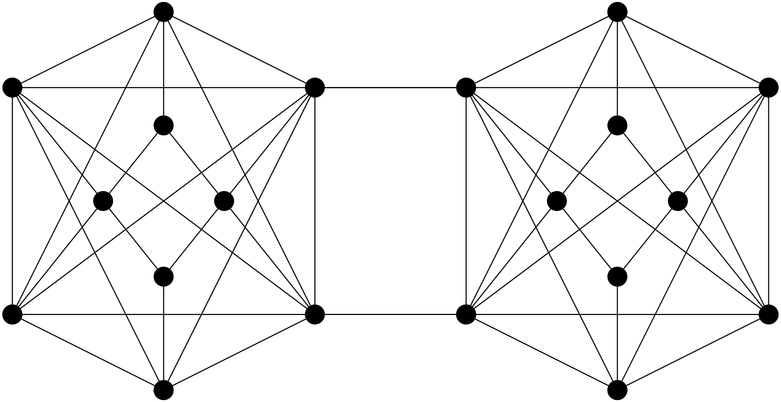
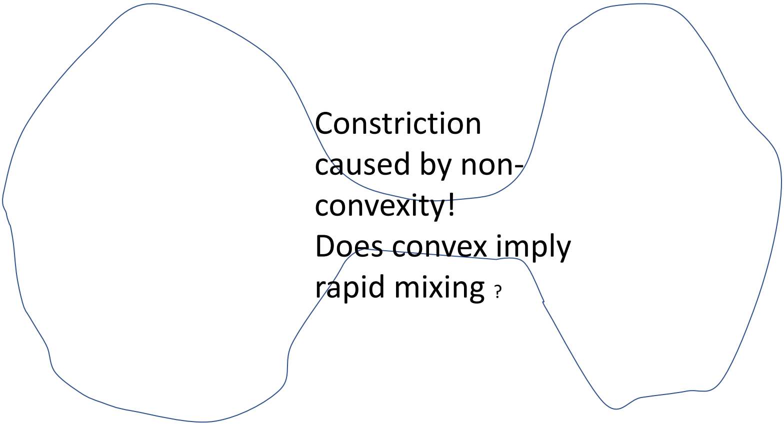


Figure: A network with a constriction. All edges have weight 1.


Constriction
caused by non-
convexity!
Does convex imply
rapid mixing ?

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- *Conductance* Φ of the MC is min over all S of $\Phi(S)$.
- **Informal Theorem** MC converges to stationarity in time $O(1/\Phi^2)$.
- Lower bound: One needs time at least $\Omega(1/\Phi)$ (Simpler than Theorem).
- Formally, for any $\varepsilon > 0$, the ε mixing time of a symmetric connected MC is min t such that $\left|\mathbf{a(t)} - \frac{1}{n}\mathbf{1}\right|_1 \leq \varepsilon$, where, $\mathbf{a(t)} = \frac{1}{t}(\mathbf{p(0)} + \mathbf{p(1)} + \cdots + \mathbf{p(t-1)})$.
- **Rapid Mixing Theorem**: The ε mixing time of a connected symmetric MC is $O(\ln n/\Phi^2\varepsilon^3)$

Idea of Proof

- Recall proof of Fundamental Theorem: Showed first that if we run one step starting from **a**(**t**), the change in the probability vector is $O(1/t)$ in l_1 norm. Let $v_i = a(t)_i/\pi_i = na_i(t)$.

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- Start with $\mathbf{a}(\mathbf{t})$, run 1 step. *Net probability flow* $h(i,j)$ from i to j: $h(i,j) = \frac{1}{n}(v_i P_{ij} - v_j P_{ji}) = \frac{1}{n}(v_i - v_j)P_{ij}$ by symmetry.

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    - So heavy vertices cannot be much heavier than the light ones (if they were, flow would be greater than $1/t$.)

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- Vertex i is **Heavy** if $v_i > 1$. Renumber so that $v_1 \geq v_2 \ldots v_{i_0} > 1 \geq v_{i_0+1} \geq \ldots v_n$. i_0 is last heavy vertex.

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- $||\mathbf{a} - (\mathbf{1}/\mathbf{n})||_1$ called **Total Variation**.

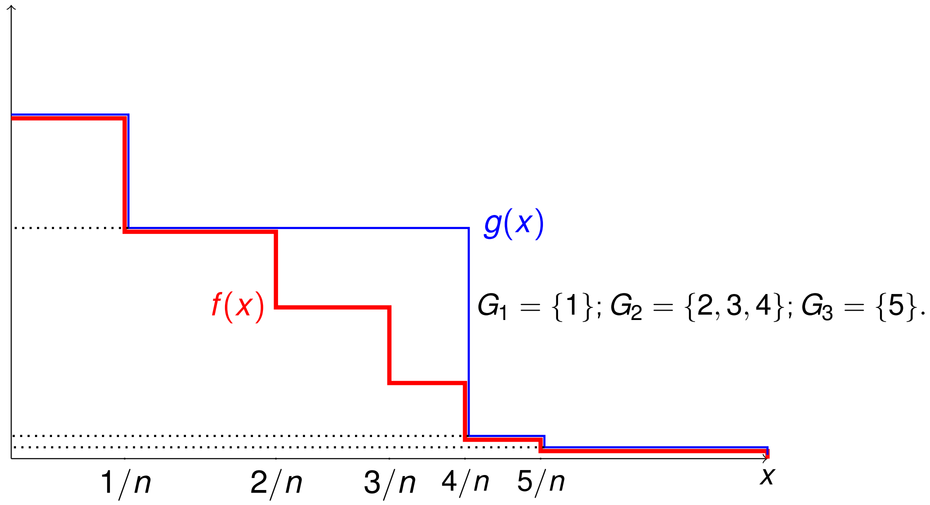

Figure: Bounding l_1 distance.

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3. But, point of groups will be: we will take as "1st set" $G_1 \cup G_2 \cup \ldots G_s$ (for each s) and "2nd set" $G_{s+2}, G_{s+3}, \ldots$. In other words, "net flow from 1st set to 2nd set" is really the "flow across" G_{s+1}. To implement the intuitive idea, we need to (i) express the total variation distance in terms of the γ above, which we do first (this is the more technical part) and (ii) prove Prob. Flow Lemma, later. (See (1)).

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6. Integrals only for intuition on “integration by parts”.
7. Now, we have TV in terms of the “heaviness gap”: $u_s - u_{s+1}$.

Probability Flow Lemma

- Definition of groups: Intuitively: If $G_1, G_2, \ldots, G_{s-1}$ are already defined, G_s has the next $\frac{\varepsilon\Phi}{4}(|G_1 \cup G_2 \cup \ldots G_{s-1}|)$ vertices. (Sizes of $|G_1 \cup G_2 \cup \ldots G_s|$ grow as $(1 + (\varepsilon\Phi/4)^s$.) More precisely,

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- **Probability Flow Lemma**: Suppose groups $G_1, G_2, \ldots, G_r, u_1.u_2, \ldots, u_r, u_{r+1}$ are as above. Then,

$$\pi(G_1 \cup G_2 \cup \ldots G_s)(u_s - u_{s+1}) \leq \frac{8}{t\Phi\varepsilon}.$$

Proof of Prob flow lemma

- This is the main lemma. Use two ways of calculating prob flow from heavy to light states when we execute one step of MC starting at $\mathbf{a}$. Prob's after that step is $\mathbf{a}P$. $\mathbf{a} - \mathbf{a}P$ is net loss of prob for each state.

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- So Net flow from A $\geq$ (Net flow from A to $\{l+1, l+2, \ldots, n\}$) $\geq \sum_{\substack{i \leq k \\ j > l}} \pi_j p_{ji}(v_i - v_j) \geq (v_k - v_{l+1}) \sum_{\substack{i \leq k \\ j > l}} \pi_j p_{ji}$.

Prob Flow Lemma-contd.

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- If $k = i_0$, the proof is similar but simpler.

Bounding number of groups

- If $\sum_{i>i_0}(1-v_i)\pi_i \leq \varepsilon$, we would be done. Wlg assume $\sum_{i>i_0}(1-v_i)\pi_i > \varepsilon$ implies $\pi(\bar{A}) \geq \varepsilon$. [Needed in last slide.]

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- Since group sizes grow geometrically, get number r of groups is at most: $r \le \ln_{1+(\varepsilon\Phi/2)}(1/\pi_1)+2 \le \ln(1/\pi_1)/(\varepsilon\Phi/2)+2$. $\pi_1 = 1/n$, so get an extra log factor.