Volume of a Convex Set

February 2, 2018

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- Recap: Alg of enclosing convex body K in a rectangular solid R and estimating proportion of random points from R (easy to draw) which fall in K does not work in general.
- But: the volume estimation problem for convex bodies in $\mathbf{R}^n$ can be reduced to (nearly) uniform sampling from convex sets in $\mathbf{R}^n$.

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- Main Result: Given these, can estimate volume of K to relative error ε in time poly in $n \ln(R/r)/\varepsilon$.

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- The ratio could be exponential in Radius of S_k / Radius of S_{k-1}.

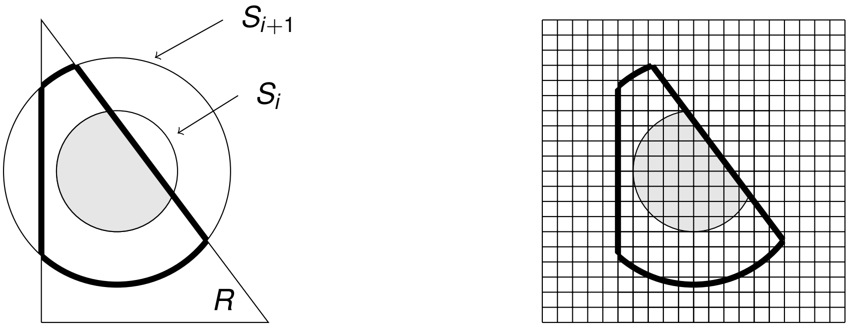

Figure: By sampling the area inside the dark line and determining the fraction of points in the shaded region we compute $\frac{Vol(S_{i+1} \cap R)}{Vol(S_i \cap R)}$.

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- So, we have give a poly time reduction of volume estimation to random sampling from a convex set.
- Don't need exact uniform sampling - approximate (in Total Variation dist) is all we know how to do by MCMC. Is that enough? Care: Now additive error. But OK, not done explicitly here.

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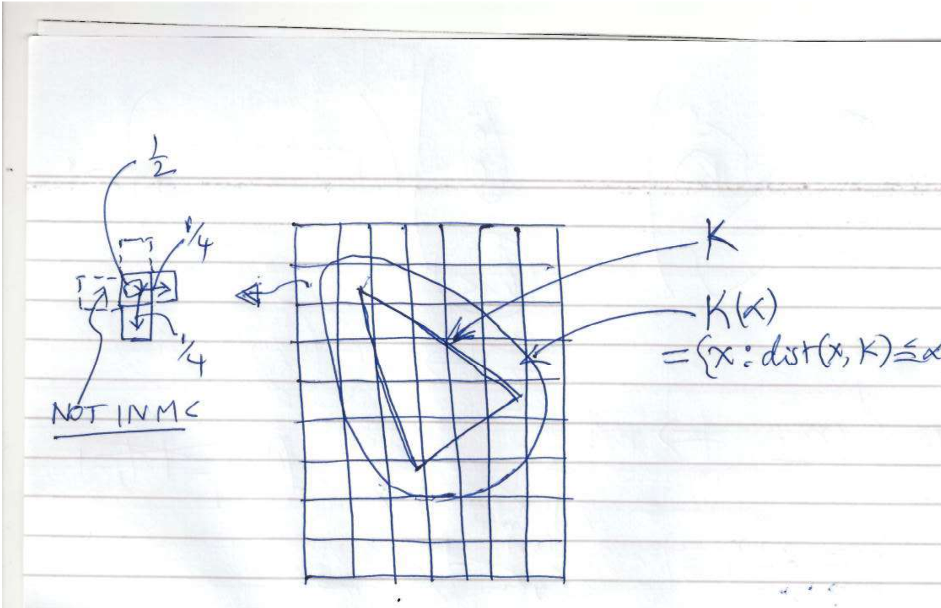
- Uniform (symmetric MC)


1/2
1/4
1/4
NOT IN MC
K
K(α)
={x : dist(x, K) ≤ α

Conductance

- Subset S of states (cubes intersecting K). Want to know:

$$\Phi(S) = \frac{\sum_{i \in S, j \in \bar{S}} \pi_i P_{ij}}{\min(\pi(S), \pi(\bar{S}))} = \frac{1}{2n} \underbrace{\frac{|(S, \bar{S})|}{\min(|S|, |\bar{S}|)}}_{\Phi'(S)}$$

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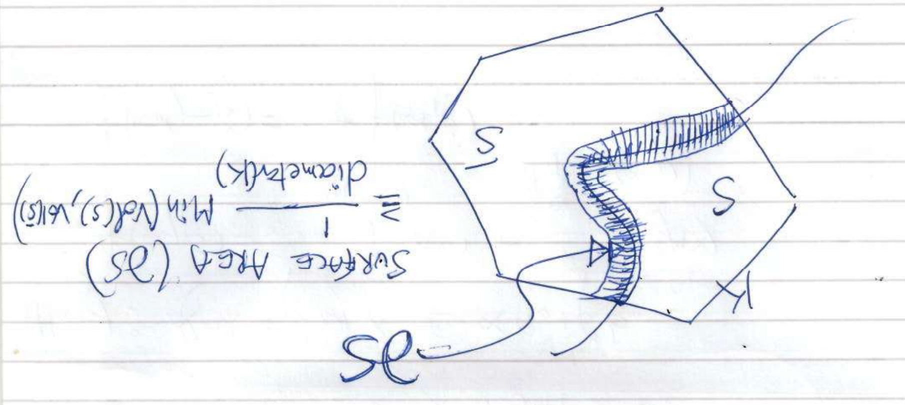
- What is best possible γ ? Can't beat $O(1/\text{diameter of } K)$. Will prove this. [Central Result for bounding Φ.]


SURFACE AREA (∂S) ≥ 1/diameter(K) Min(Vol(S), Vol(S̄))
S
S̄
K
∂S

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- Isoperimetry Restated: For any partition of a convex into three pieces: S_1, S_2, S_3 with $\text{dist}(x, y) \geq \varepsilon \forall x \in S_1, y \in S_3$, prove $\text{Vol}(S_2) \geq \gamma\varepsilon\text{min}(\text{Vol}(S_1), \text{Vol}(S_3))$.

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- The central part of the proof of Isoperimetry is to reduce this to a 1-dimensional case and then prove (for contradiction) that for any 1-d case, these two inequalities cannot hold.

Reduction to “Needle-like” case

- **Localization Lemma** Suppose K is a convex body and g, h are lower semi-continuous functions on K with $\int_K g, \int_K h > 0$. Then for any $\varepsilon > 0$, (think ε infinitesimal) there is a convex set $P \subseteq K$ and points $a, b \in P$ such that

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- Idea of proof: Use (simple version of) Borsuk-Ulam theorem to cut the domain of integration repeatedly, preserving $\int g, \int h > 0$

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- Once we have K_i, we choose a half-space H such that it bisects both $\int_{K_i} g$ and $\int_{K_i} h$ and set $K_{i+1} = K_i \cap H$.
- Borsuk-Ulam guarantees that given any 2-dim subspace V, we can choose such an H whose normal lies in V.
- Choose V to be the space spanned by the two largest axes of the min vol ellipsoid containing K_i. This process will ensure that no two axes of K_i can both remail long as $i \to \infty$, hence, K_i become needle-like.

Proof for the needle

- **Claim** If P is a convex body with $P \subseteq \varepsilon-$neighbouhood of line segment $a \to b$, where, $\varepsilon \to 0$ and let $p(x)$ be the cross-sectional area of P at $x \in ab$. Suppose S_1, S_2, S_3 is a partition of ab with $\text{dist}(x, y) \geq \delta \forall x \in S_1, y \in S_3$. Then, $\int_{x \in (ab) \cap S_2} p(x) \geq \min(\int_{x \in S_1} p(x), \int_{x \in S_3} p(x) / \text{dia}(P) - \varepsilon'$, where $\varepsilon' \to 0$.

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- Follows from the fact that the cross-sectional area $p(x)$ is unimodal - i.e., does not decrease and then increase. So, for any $x \in (ab) \cap S_2$, $p(y) \leq p(x)$ either for all $y > x$ or for all $y < x$.

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  - **Inertial Ellipsoid** Moment-of-inertia: Matrix M with $M_{ij} = E_K(x_i x_j)$. For $\tau = M^{-1/2}$, $E_{\tau K}(x_i x_j) = B$.
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