E0 337: Topics in Advanced Cryptography	4/Sep/23
Lecture 4	
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1 Welcome to E0 337

1.1 Course Information

Contact information and office hours:

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1.2 Course Topics

The course will cover the following topics:

- Bayesian Games
- Auctions
- Incentive compatibility

2 Introduction

Mechanism design is a field in game theory that focuses on creating systems with specific goals. Here we focus on Bayesian games, Bayesian equilibrium, and auctions, like first-price and second-price auctions. The key challenge is ensuring incentive compatibility, either through DSIC (Dominant Strategy Incentive Compatibility) or BIC (Bayesian Incentive Compatibility), making it advantageous for participants to reveal their true preferences, even in uncertain situations. These concepts play a vital role in designing strategic systems.

3 Bayesian Games

Bayesian games are a class of strategic games in the field of game theory where players have incomplete information about certain aspects of the game. In a Bayesian game, each player's uncertainty is modeled using probability distributions over the possible types or characteristics of the other players. These probability distributions capture the players' beliefs about the likelihood of different types of opponents or their private information. In essence, Bayesian games extend traditional game theory to situations where players not only make decisions based on their preferences and strategies but also consider their probabilistic beliefs about the characteristics or information held by other players. This modeling framework allows for a more realistic representation of strategic interactions in situations where players have varying degrees of uncertainty about their opponents.
let us define bayesian games definition is taken from [1]

Definition 1. A Bayesian game, which is a strategic form game with incomplete information, is defined as

$$\Gamma^b = \langle N, (S_i)_{i\in N}, (T_i)_{i\in N}, (p_i)_{i\in N}, (u_i)_{i\in N} \rangle$$

- $N = \{1, 2, ..., n\}$ is a set of players;
- $S_1, S_2, ..., S_n$ are sets called the strategy sets of the players 1, . . . , n, respectively;
- T_i is the set of types of player i where $i = 1, 2, ..., n$.
- p_i: $T_i \to \Delta T_{-i}$ is the probability function for player i
- $u_i : S_1 \times S_2 \times \ \times S_n \to \mathbb{R}$ for i = 1, 2, . . . , n are mappings called the utility functions or payoff functions.

Example 1. *Let us consider a bargaining game[1]: Player 1 sells an object, and player 2 is the potential buyer. Both players know their own object value but have uncertain beliefs about the other's value, ranging from 1 to 100 with equal probability. They simultaneously propose bids between 0 and 100. If the buyer's bid is higher or equal to the seller's bid, they trade at the average bid price; otherwise, no trade happens. Both players are risk-neutral, so their utility is equivalent to the monetary profit from the trade. This game can be formulated as a Bayesian game as follows:*

$$\begin{aligned}
N &= \{1, 2\} \\
T_i &= \{1, 2, \dots, 100\} \\
S_i &= \{1, 2, \dots, 100\} \\
p_i(t_{-i}|t_i) &= \frac{1}{100} \\
u_1(s, t) &= \begin{cases} \frac{s_1+s_2}{2} - t_1 & \textit{if } s_2 \geq s_1 \\ 0 & \textit{otherwise} \end{cases} \\
u_2(s, t) &= \begin{cases} t_2 - \frac{s_1+s_2}{2} & \textit{if } s_2 \geq s_1 \\ 0 & \textit{otherwise} \end{cases}
\end{aligned}$$

here the beliefs p1 and p2 are consistent with the prior :

$$p(t) = \frac{1}{10000}, \quad \forall t \in T = \{1, 2, \dots, 100\} \times \{1, 2, \dots, 100\}$$

3.1 Bayesian Nash Equilibrium

A Bayesian Nash Equilibrium is a concept that extends the traditional Nash Equilibrium to situations where players have incomplete information about the game or the strategies of other players. The following definition is from [1]

Definition 2. A pure strategy Bayesian Nash equilibrium in a Bayesian game

$$\Gamma^b = \langle N, (S_i), (T_i), (p_i), (u_i) \rangle$$

can be defined in a natural way as a pure strategy Nash equilibrium of the equivalent game. That is, a profile of strategies $(s_1^*, ...s_n^*)$ is a pure strategy Bayesian Nash equilibrium if $\forall i \in N$; $\forall s_i : T_i \to S_i$; $\forall t_i \in T_i$

$$u_i((s_i^*, s_{-i}^*)|t_i) \geq u_i((s_i, s_{-i}^*)|t_i)$$

4 Auctions

Auctions are commonly used in various online transactions and web-based applications, serving as practical examples of mechanisms involving money. Mechanism design theory offers a systematic approach to design auctions that guarantee specific desirable properties. In this context, we explore various auction types and fundamental mechanism design considerations. Specifically, we focus on three primary types of auctions for single, indivisible items[3]: the English auction , the first-price auction, and the second-price auction (also known as the Vickrey auction).
Auctions include the following sets :

- $N = \{1, 2, ...n\}$ set of bidders
- $V = \{v_1, v_2, ...v_n\}$ valuation of the auctioned item of each bidder
- $b = \{b_1, b_2, ...b_n\}$ bids of each bidder

4.1 English Auction

An English auction, also known as an ascending bid auction, is a type of auction in which the price for an item or service starts low and gradually increases as participants bid higher amounts. It is a public and open auction format where participants openly compete to place the highest bid. The auctioneer or the seller begins by announcing a minimum acceptable bid, and interested participants sequentially raise their bids. The bidding continues until no one is willing to place a higher bid, at which point the highest bidder wins the item and is obligated to pay the final amount they bid.
Key features of English auctions include:

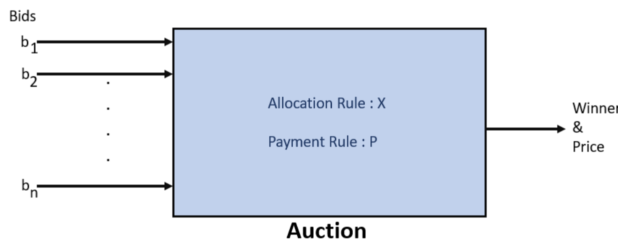
- **Transparent Bidding:** Bidders can see each other's bids, fostering a competitive atmosphere.
- **Bidding Increments:** The auctioneer or auction platform typically sets the amount by which bids must increase (bid increments) for each round of bidding.
- **Dynamic Pricing:** The final price is determined by the highest bid when bidding concludes. This means that the winning bidder pays the maximum amount they were willing to bid.

English auctions are commonly used for selling various items, including art, antiques, real estate, and even online products in e-commerce platforms. They encourage competitive bidding and often result in a fair market price for the item being auctioned.

4.2 Second Price Auction (Vickrey Auction)

A second-price auction, also known as a Vickrey auction after its developer William Vickrey, is a type of sealed-bid auction where participants submit private bids for an item or service without knowing the bids of other participants in a second-price auction.

- **Bidding Process:** Each participant submits a sealed bid representing the maximum amount they are willing to pay for the item.
- **Allocation Rule:** The participant who submitted the highest bid wins the item.
- **Payment Rule :**the second-highest bid. In other words, the winning bidder pays the price of the highest losing bid.

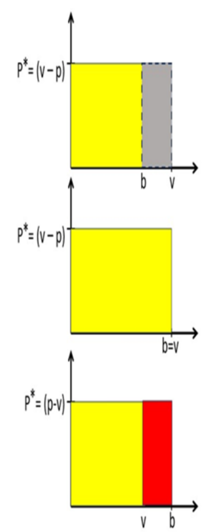


$$i^* = \underset{i \in n}{\operatorname{argmax}}\, b_i$$

$$p_i^* = \max_{i \neq i^*} b_i$$

Now, let's analyze the equilibrium of second-price auctions for a specific bidder, i.
Here we have three cases:
The utility of the bidders is defined as :

$$u_i = v_i - p^*$$



Case 1 : $v_i < b_i$
In this case, the utility of the bidder i is as follows:

$$u_i = \begin{cases} v_i - p^* & \text{if } i \text{ wins} \\ 0 & \text{Otherwise} \end{cases}$$

Case 2 :$v_i = b_i$
In this case, the utility of the bidder i is as follows:

$$u_i = \begin{cases} v_i - p^* & \text{if } i \text{ wins} \\ 0 & \text{Otherwise} \end{cases}$$

Case 3 :$v_i > b_i$
In this case, the utility of the bidder i is as follows:

$$u_i = \begin{cases} p^* - v_i & \text{if } i \text{ wins} \\ 0 & \text{Otherwise} \end{cases}$$

In each case 1 & 2, the utility $u_i = v_i - p^*$ is the maximum utility for bidder i, so $v_i = b_i$ is the dominant strategy for bidder i as he won't get any incentive by not bidding his true valuation. However in case 3, if the bidder i wins, then his utility is $u_i = p^* - v_i$, which is negative; else he gets 0, so in this case the maximum utility for bidder i is 0. i.e., truth-telling is the dominant strategy this auction captures (DSIC).

4.3 First price auctions

In this scenario, prospective buyers submit sealed bids, and the item is awarded to the bidder with the highest bid. In cases where multiple bidders submit the highest bids, a predefined tie-breaking rule is employed to determine the ultimate winner. The winning bidder must then pay the exact price they originally bid for the item.

- **Bidding Process:** Each participant submits a sealed bid representing the maximum amount they are willing to pay for the item.

- **Allocation Rule:** The participant who submitted the highest bid wins the item.
- **Payment Rule :**The winner pays the highest bid that he has submitted.

Now, let's analyze the equilibrium of First-price auctions for that, we are considering the following conditions on the valuations and the probabilities of the valuations. for two bidders $\{v_1, v_2\} \in [0,1]$ and their probabilities $\{\alpha_1, \alpha_2\} \in [0,1]$ also lets say $b_1 = \alpha_1 v_1$ and $b_2 = \alpha_2 v_2$ Then, we can define the expected utility of bidder 1 as:

$$\begin{aligned}
\mathbb{E}[u_1] &= (v_1 - b_1)\mathbb{P}r[\alpha_2 v_2 < b_1] + 0 * \mathbb{P}r[\alpha_2 v_2 > b_1] \\
&= (v_1 - b_1)\frac{b_1}{\alpha_2} \\
\max \mathbb{E}[u_1] &= \max(v_1 - b_1)\frac{b_1}{\alpha_2} \qquad (1)
\end{aligned}$$

Now, to get the maximum expected value let's solve eq(1) by differentiating with respect to b_1 and equating to zero.

$$\begin{aligned}
\frac{\partial}{\partial b}\left[E[u_1]\right] &= \frac{v_1 b_1}{\alpha_2} - \frac{b_1^2}{\alpha_2} = 0 \\
b_1 &= \frac{v_1}{2}
\end{aligned}$$

the best bid for both bidders is bidding $\frac{v_i}{2}$ this is a Bayesian Nash equilibrium. we can formulate this for n bidders is well lets consider n bidders $\{v_1, v_2, ..., v_n\} \in [0,1]$ then their best bid set is given as $\{(\frac{n-1}{n}v_1), (\frac{n-1}{n}v_2), ..., (\frac{n-1}{n}v_n)\}$

5 Incentive Compatibility

Mechanism design addresses the challenge of collecting truthful information from agents about their preferences and characteristics. To ensure honesty, the approach is to create incentives that make truth-telling the best strategy while respecting rationality. This is achieved through incentive compatibility, which comes in two forms: Dominant Strategy Incentive Compatibility (DSIC), where truth-telling is the best strategy for each agent regardless of others' actions, and Bayesian Nash Incentive Compatibility (BIC), where truth-telling is the best strategy for each agent given their expectations of others' types.

5.1 Dominant Strategy Incentive Compatibility (DSIC)

DSIC ensures that it is in an individual's best interest to be completely honest about their preferences and information during the auction or mechanism. They don't need to consider what others might do; they should always find it most advantageous to tell the truth. DSIC mechanisms are highly desirable in situations where trust and honesty among participants are essential, as they provide strong incentives for truthful behavior and reduce the potential for strategic manipulation or deception. The definition of DSIC [2]

Definition 3. Dominant Strategy Incentive Compatibility (DSIC)
A social choice function $f\colon T_1 \times T_2 ... \times T_n \to X$ is said to be a Dominant Strategy Incentive Compatibility if the direct revelation mechanism $\mathcal{D} = ((T_i)_{i \in N}, f(.))$has a weakly dominated strategy equilibrium $s^*(.) = (s_1^*(.), s_2^*(.), ...s_n^*(.))$ in which $s_{i(t_i)}^* = t_i, \forall t_i \in T_i, \forall i \in N$.

Example 2. *The second price auction that we have discussed in section [4.2] achieves DSIC*

5.2 Bayesian Incentive Compatibility (BIC)

In BIC, participants take into account not only their own preferences and information but also their beliefs or expectations about the preferences and information of other participants. Despite this uncertainty, participants still find it in their best interest to be truthful about their own information. BIC ensures that participants are incentivized to honestly reveal their information, taking into account the fact that they may not know for sure what others know or will do. This concept is particularly valuable in settings where participants have incomplete information about each other but still want to make strategic decisions that maximize their own utility. BIC mechanisms are designed to encourage honest behavior even in such uncertain environments. The definition of BIC[2]

Definition 4. Bayesian Incentive Compatibility (BIC)
A social choice function $f : T_1 \times ... \times T_n \to X$ is said to be Bayesian incentive compatible (or truthfully implementable in Bayesian Nash equilibrium) if the direct revelation mechanism $\mathcal{D} = ((T_i)_{i\in N}, f(.))$ has a Bayesian Nash equilibrium $s^*(.) = (s_1^*(.), ..., s_n^*(.))$ in which $s_i^*(t_i) = t_i, \forall t_i \in T_i, \forall i \in N$.

Example 3. *The First-price auction with a modified payment rule allocation is the same as the traditional first-price auctions, but the payment by the winning buyer is equal to half of his bid. Let's analyze the expected utilities of bidders by taking same assumptions that we have used in first-price auctions, let's say bidder two is bidding his true valuation v_2 then the expected utility of bidder one is given by*

$$\begin{aligned} \mathbb{E}[u_1] &= (v_1 - \frac{b_1}{2})\mathbb{P}r[\alpha_2 v_2 < b_1] + 0 * \mathbb{P}r[\alpha_2 v_2 > b_1] \\ &= (v_1 - \frac{b_1}{2})\frac{b_1}{\alpha_2} \\ \max \mathbb{E}[u_1] &= \max(v_1 - \frac{b_1}{2})\frac{b_1}{\alpha_2} \qquad (2) \end{aligned}$$

Now, to get the maximum expected value, let's solve eq(2) by differentiating with respect to b_1 and equating to zero.

$$\begin{aligned} \frac{\partial}{\partial b}[E[u_1]] &= \frac{v_1 b_1}{\alpha_2} - \frac{b_1^2}{2\alpha_2} = 0 \\ b_1 &= v_1 \end{aligned}$$

Thus it is optimal in expectation for bidder 1 to reveal his true private value if bidder 2 reveals his true value. The same situation applies to bidder 2. Thus, a best response for each bidder, given that the other bidder truthfully reports his true type, is to report his true type. This captures (BIC)

References

[1] R. B. Myerson. Game theory - analysis of conflict. 1991.

[2] Y. Narahari. *Game Theory And Mechanism Design, chapter 16.* Iisc Lecture Notes Series. World Scientific Publishing Company, 2014.

[3] Y. Narahari. *Game Theory And Mechanism Design chapter 20.* Iisc Lecture Notes Series. World Scientific Publishing Company, 2014.