Chomsky Normal Form for Context-Free Gramars

Deepak D'Souza

Department of Computer Science and Automation
Indian Institute of Science, Bangalore.

20 September 2013

Outline

1. CNF
2. Converting to CNF
3. Correctness

Chomsky Normal Form

A Context-Free Grammar G is in Chomsky Normal Form if all productions are of the form

$$\begin{aligned} X &\rightarrow YZ \text{ or} \\ X &\rightarrow a \end{aligned}$$

Its a "normal form" in the sense that

CNF

Every CFG G can be converted to a CFG G' in Chomsky Normal Form, with $L(G') = L(G) - \{\epsilon\}$.

Example

CFG G_4

$$S \rightarrow (S) \mid SS \mid \epsilon.$$

"Equivalent" grammar in CNF:

CFG G_4' in CNF

$$\begin{aligned} S &\rightarrow LX \mid SS \mid LR \\ X &\rightarrow SR \\ L &\rightarrow (\\ R &\rightarrow) \end{aligned}$$

Why is CNF useful?

- Gives us a way to do parsing: Given CFG G and $w \in A^*$, does $w \in L(G)$?

Why is CNF useful?

- Gives us a way to do parsing: Given CFG G and $w \in A^*$, does $w \in L(G)$?
    - If G is in CNF, then the length of derivation of w (if one exists) can be bounded by $2|w|$.

Why is CNF useful?

- Gives us a way to do parsing: Given CFG G and $w \in A^*$, does $w \in L(G)$?
  - If G is in CNF, then the length of derivation of w (if one exists) can be bounded by $2|w|$.
- Makes proofs of properties of CFG's simpler.

Procedure to convert a CFG to CNF

- Main problem is “unit” productions of the form $A \longrightarrow B$ and ϵ-productions of the form $B \longrightarrow \epsilon$.
- Once these productions are eliminated, converting to CNF is easy.

Procedure to remove unit and ϵ-productions

Given a CFG $G = (N, A, S, P)$.

- Repeatedly add productions according to the steps below till no more new productions can be added.
  1. If $A \to \alpha B \beta$ and $B \to \epsilon$ then add the production $A \to \alpha\beta$.
  2. If $A \to B$ and $B \to \gamma$ then add the production $A \to \gamma$.
- Let resulting grammar be $G' = (N, A, S, P')$.
- Let G'' be grammar (N, A, S, P''), where P'' is obtained from P' by dropping unit- and ϵ-productions.
- Return G''.

Example

Apply procedure to the grammar below:

CFG G_4

$$S \rightarrow (S) \mid SS \mid \epsilon.$$

Correctness claims

- Algorithm terminates.

Correctness claims

- Algorithm terminates.
  - Notice that each new production added has a RHS that is a subsequence of RHS of an original production in P.

Correctness claims

- Algorithm terminates.
  - Notice that each new production added has a RHS that is a subsequence of RHS of an original production in P.
- G' generates the same language as G.
  - Let G'_i be grammar obtained after i-th step, with $G'_0 = G$.
  - Then clearly $L(G'_{i+1}) = L(G'_i)$.

Correctness of G''

Claim

$L(G'') = L(G) - \{\epsilon\}$.

Subclaim

Let $w \in L(G')$ with $w \neq \epsilon$. Then any minimal-length derivation of w in G' does not use unit or ϵ-productions.

Proof of Subclaim

Subclaim

Let $w \in L(G')$ with $w \neq \epsilon$. Then any minimal-length derivation of w in G' does not use unit or ϵ-productions.

Consider a derivation of w in G' which uses a production $B \longrightarrow \epsilon$. It must be of the form

$$S \stackrel{l}{\Rightarrow} \alpha X \beta \stackrel{1}{\Rightarrow} \alpha\gamma B\delta\beta \stackrel{m}{\Rightarrow} \alpha'\gamma' B\delta'\beta' \stackrel{1}{\Rightarrow} \alpha'\gamma'\delta'\beta' \stackrel{n}{\Rightarrow} w.$$

Proof of Subclaim

Subclaim

Let $w \in L(G')$ with $w \neq \epsilon$. Then any minimal-length derivation of w in G' does not use unit or ϵ-productions.

Consider a derivation of w in G' which uses a production $B \longrightarrow \epsilon$. It must be of the form

$$\begin{array}{llllll} S & \stackrel{l}{\Rightarrow} \alpha X \beta & \stackrel{1}{\Rightarrow} \alpha\gamma B\delta\beta & \stackrel{m}{\Rightarrow} \alpha'\gamma' B\delta'\beta' & \stackrel{1}{\Rightarrow} \alpha'\gamma'\delta'\beta' & \stackrel{n}{\Rightarrow} w. \\ S & \stackrel{l}{\Rightarrow} \alpha X \beta & \stackrel{1}{\Rightarrow} \alpha\gamma\delta\beta & \stackrel{m}{\Rightarrow} \alpha'\gamma'\delta'\beta' & & \stackrel{n}{\Rightarrow} w. \end{array}$$

Now consider a derivation of w in G' which uses a production $A \longrightarrow B$. It must be of the form

$$S \stackrel{l}{\Rightarrow} \alpha A \beta \stackrel{m}{\Rightarrow} \alpha' A \beta' \stackrel{1}{\Rightarrow} \alpha' B \beta' \stackrel{n}{\Rightarrow} \alpha'' B \beta'' \stackrel{1}{\Rightarrow} \alpha''\gamma\beta'' \stackrel{p}{\Rightarrow} w.$$

Proof of Subclaim

Subclaim

Let $w \in L(G')$ with $w \neq \epsilon$. Then any minimal-length derivation of w in G' does not use unit or ϵ-productions.

Consider a derivation of w in G' which uses a production $B \longrightarrow \epsilon$. It must be of the form

$$
\begin{array}{llllll}
S \overset{l}{\Rightarrow} \alpha X \beta & \overset{1}{\Rightarrow} \alpha\gamma B\delta\beta & \overset{m}{\Rightarrow} \alpha'\gamma' B\delta'\beta' & \overset{1}{\Rightarrow} \alpha'\gamma'\delta'\beta' & \overset{n}{\Rightarrow} w. \\
S \overset{l}{\Rightarrow} \alpha X \beta & \overset{1}{\Rightarrow} \alpha\gamma\delta\beta & \overset{m}{\Rightarrow} \alpha'\gamma'\delta'\beta' & & \overset{n}{\Rightarrow} w.
\end{array}
$$

Now consider a derivation of w in G' which uses a production $A \longrightarrow B$. It must be of the form

$$
\begin{array}{lllllll}
S \overset{l}{\Rightarrow} \alpha A \beta & \overset{m}{\Rightarrow} \alpha' A \beta' & \overset{1}{\Rightarrow} \alpha' B \beta' & \overset{n}{\Rightarrow} \alpha'' B \beta'' & \overset{1}{\Rightarrow} \alpha''\gamma\beta'' & \overset{p}{\Rightarrow} w. \\
S \overset{l}{\Rightarrow} \alpha A \beta & \overset{m}{\Rightarrow} \alpha' A \beta' & \overset{1}{\Rightarrow} \alpha'\gamma\beta' & \overset{n}{\Rightarrow} \alpha''\gamma\beta'' & & \overset{p}{\Rightarrow} w.
\end{array}
$$