**Automata Theory and Computability**

*Assignment 1*

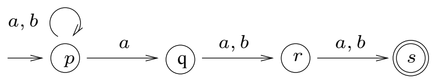
(Due on Fri 06 Sep 2013)

1. Give a DFA for the language of all strings over the alphabet $\{0, 1\}$ which contain an occurrence of 010 but *not* 100 as a contiguous substring.

2. Consider the language of all strings over the alphabet $\{a, b\}$ which satisfy the property that in every prefix the difference between the number of $a$'s and $b$'s is at most 2. Thus, $aabab$ is in the language, while $abaaab$ is not. Is this language regular? Justify your answer.

3. Give a DFA/NFA for all strings in $\{0, 1\}^*$ which represent (in binary) numbers which leave a remainder of 1 on dividing by 3.

4. Consider the NFA below:




   (a) Use the subset construction to obtain an equivalent DFA for the NFA below. Label each state of the DFA with the subset of states of the NFA that it corresponds to.

   (b) Give an 8 state DFA which accepts the same language.

5. This question asks you to formalize the subset construction and prove its correctness.

   (a) Define formally the "subset automaton" $\mathcal{S}_{\mathcal{A}}$ for a given NFA $\mathcal{A} = (Q, S, \Delta, F)$ over an alphabet $\Sigma$.

   (b) Recall that we had defined the relation $\longrightarrow^*$ corresponding to the NFA $\mathcal{A}$ in class as follows:

$$p \xrightarrow{\epsilon}^* p$$
$$p \xrightarrow{wa}^* q \quad \text{iff} \quad \exists r \text{ such that } p \xrightarrow{w}^* r \text{ and } q \in \Delta(r, a).$$

   Prove formally that the transition function $\delta$ of the subset automaton satisfies the property:

$$\widehat{\delta}(X, w) = \{q \mid \exists p \in X : \ p \xrightarrow{w}^* q\}.$$

   (c) Use the claim above to conclude that the languages accepted by the NFA $\mathcal{A}$ and its subset automaton $S_{\mathcal{A}}$ coincide.

6. Let $L$ and $M$ be regular languages over alphabets $\Sigma_1$ and $\Sigma_2$ respectively. Prove that the following languages are also regular. Describe your construction formally. There is no need to prove your construction correct.

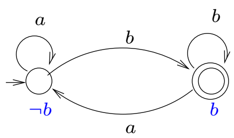
   - $L^*$ (Recall that $L^*$ is defined to be $\{\epsilon\} \cup L \cup L^2 \cup L^3 \cup \cdots$, where $L^n = \{w_1 \cdots w_n \mid w_i \in L \text{ for each } i\}$.)

- $L \parallel M$, where $L \parallel M$ is the "shuffle" of $L$ and $M$, defined as follows. Let $\Sigma = \Sigma_1 \cup \Sigma_2$. For a string $w$ in $\Sigma^*$ we define the projection of $w$ to $\Sigma_1$, written $w{\restriction}\Sigma_1$, to be the string in $\Sigma_1$ obtained from $w$ by erasing all letters *not in* $\Sigma_1$. Thus if $\Sigma_1$ is $\{a, b\}$ and $\Sigma_2$ is $\{b, c\}$, then $bcabac{\restriction}\Sigma_1 = baba$. We can now define $L \parallel M$ as

$$L \parallel M = \{w \in \Sigma^* \mid w{\restriction}\Sigma_1 \in L \text{ and } w{\restriction}\Sigma_2 \in M\}.$$

7. (a) Construct a regular expression that describes the lanaguage accepted by the DFA below, using Kleene's construction (using $L^X_{pq}$'s).

   (b) Describe the language accepted using the equation solving method done in class. For this first write down the equations induced by the DFA. Then use the matrix-based expression to describe the least solution to the equations.




8. Let $\mathcal{A} = (Q, S, \Delta, F)$ be an NFA. Consider the system of equations (Eq) induced by $\mathcal{A}$ below:

$$x_p = \begin{cases} \bigcup_{q \in \Delta(p,a)} (\{a\} \cdot x_q) \ \cup \{\epsilon\} & \text{if } p \in F, \\ \bigcup_{q \in \Delta(p,a)} (\{a\} \cdot x_q) & \text{otherwise.} \end{cases}$$

   Show that (Eq) has a *unique* solution, given by $x_p = L_p$, where

$$L_p = \{w \in \Sigma^* \mid \exists f \in F : \ p \xrightarrow{w}{}^* f\},$$

   for each $p \in Q$.

9. Let $L$ be a regular set over an alphabet $\Sigma$. Which of the following are regular? Justify your answers.

   (a) $\mathit{suff}(L) = \{v \mid \exists u \in \Sigma^* : uv \in L\}$

   (b) $\mathit{mid\text{-}thirds}(L) = \{v \mid \exists u, w : |u| = |v| = |w| \text{ and } uvw \in L\}$.

10. Let $L$ be a regular set. Show that the set

$$DM_L = \{xz \mid \exists y, |x| = |y| = |z|, xyz \in L\}$$

   got by deleting the middle thirds of $L$ is not regular. (Hint: Use the fact that regular sets are closed under intersection and choose appropriate sets for intersection to get a contradiction).