Homomorphisms

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Outline

1. Homomorphisms
2. Two results regarding homomorphisms
3. Some applications
4. Proof of results

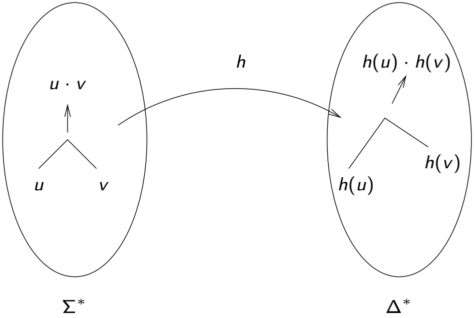
Homomorphism from Σ^* to Δ^*

A map

$$h : \Sigma^* \to \Delta^*$$

satisfying:

1. $h(\epsilon) = \epsilon$.
2. $h(u \cdot v) = h(u) \cdot h(v)$.



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  - That is $h(\epsilon) = h(\epsilon) \cdot h(\epsilon)$.
  - Only ϵ can satisfy this equation.
- h is determined completely by h on Σ.
  - For example above: h is determined by

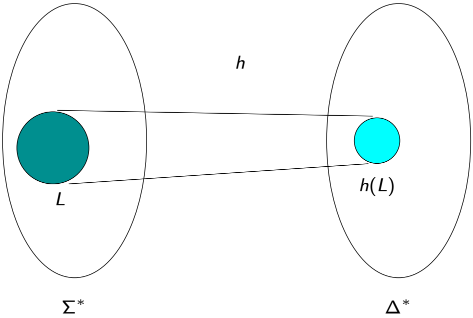
$$\begin{array}{ccc} a & \mapsto & a \\ b & \mapsto & \epsilon. \end{array}$$

Homomorphisms preserve regular sets

Let $h : \Sigma^* \to \Delta^*$ be a homomorphism. Then

Fact 1

If $L \subseteq \Sigma^*$ is regular, then so is $h(L)$.

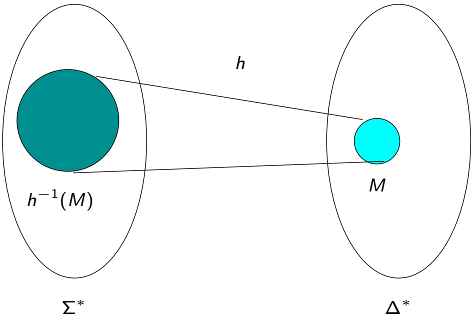


Homomorphisms preserve regular sets

- Define $h^{-1} : \Delta^* \to 2^{\Sigma^*}$ by $h^{-1}(w) = \{u \in \Sigma^* \mid h(u) = w\}$.
- Define $h^{-1}(M) = \bigcup_{w \in M} h^{-1}(w)$.

Fact 2

If $M \subseteq \Delta^*$ is regular, then so is $h^{-1}(M)$.



Some applications: NFA's with ϵ-transitions

- NFA with ϵ-labelled transitions:

$$\Delta : Q \times (A \cup \{\underline{\epsilon}\}) \rightarrow 2^Q.$$

- Accepts a word w if there is a path from a start state to a final state labelled by a word u such that $h(u) = w$, where

$$\begin{array}{rcl} a & \mapsto & a \\ \underline{\epsilon} & \mapsto & \epsilon. \end{array}$$

- Claim: NFA's with ϵ-transitions accept only regular sets.
- Observe that $L(\mathcal{A}) = h(L(\mathcal{A}'))$, where $\mathcal{A}'$ is $\mathcal{A}$ viewed as running over the alphabet $A \cup \{\underline{\epsilon}\}$.

A wrong application

- Consider homomorphism h on $\{a, b\}^*$, given by:

$$\begin{array}{ccc} a & \mapsto & a \\ b & \mapsto & a. \end{array}$$

- Let $L = \{a^n b^n \mid n \geq 0\}$.
- Then $h(L) = \{a^{2n} \mid n \geq 0\}$.
- Can we conclude that L is regular?

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Construct DFA for $h^{-1}(M)$ using DFA $\mathcal{A}$ for M.