Pumping Lemma and Ultimate Periodicity

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Outline

1. Pumping Lemma
2. Ultimate Periodicity
Two necessary conditions for regularity

- **Pumping Lemma**: Any “long enough” word in a regular language must have a “pump.”
- Lengths of words in a regular language are “ultimately periodic.”
Pumping lemma for regular languages

Based on a simple observation:
In a given a DFA $A$, if a path $p$ in it is longer than the number of states in $A$ then $p$ must have a loop in it.

So if $uvw$ is accepted along this path, then so is $uw$, $uv^2w$, ....
Pumping Lemma statement

For any regular language $L$ there exists a constant $k$, such that for any word $t \in L$ of the form $xyz$ with $|y| \geq k$, there exist strings $u$, $v$, $w$ such that:

1. $y = uvw$, $v \neq \epsilon$, and
2. $xuv^iwz \in L$, for each $i \geq 0$. 

Pumping Lemma
## Game induced by $L$

A play in $G_L$:

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<th>You</th>
</tr>
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<td>Provides a $k$.</td>
<td>Choose $t \in L$, with decomposition $x, y, z$, and $</td>
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<tr>
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Demon wins the play if $xuv^i wz \in L$, otherwise You win.

- If $L$ is regular then Demon has a winning strategy in $G_L$.
- Equivalently: If You have a winning strategy in $G_L$, then $L$ is not regular.
Pumping Lemma is *not* a sufficient condition for regularity.

- There exist non-regular languages $L$ for which the Demon has a winning strategy in $G_L$. 
Example applications of Pumping Lemma

Describe Your strategy to beat the Demon in the games for:

- \( \{a^n b^n \mid n \geq 0\} \).
- \( \{a^{2n} \mid n \geq 0\} \).
Two problems to think about

1. If $L \subseteq \{a\}^*$, show that $L^*$ is regular.
2. Show that there exists a language $L \subseteq A^*$ such that neither $L$ nor its complement $A^* - L$ contains an infinite regular set.
A subset \( X \) of \( \mathbb{N} \) is **ultimately periodic** if

- There exist \( n_0 \geq 0, \ p \geq 1 \) in \( \mathbb{N} \), such that for all \( m \geq n_0 \),
  \[
  m \in X \ \text{iff} \ m + p \in X.
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- Or equivalently: $X = F \cup A_1 \cup \cdots \cup A_k$, for some finite set $F$ and arithmetic progressions $A_i$ of same period $p$. 
Examples of u.p. sets

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- \{10, 12, 14, 16, \ldots\} \cup \{5, 10, 15, \ldots\} is u.p.
- \{0, 2, 4, 8, 16, 32, \ldots\} is not u.p.
Ultimate Periodicity of Regular Languages

Claim

If $L$ is a regular language then $\text{lengths}(L)$ is an ultimately periodic set.

Proof:

- Argue for language over single-letter alphabet.
- Infer for general language.
What does a DFA on single-letter alphabet look like?

\[
\text{lengths}(L(A)) = \{2\} \cup \{5, 11, 17, \ldots\} \cup \{8, 14, 20, \ldots\}.
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