Regular Expressions

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Outline

1. Regular Expressions
2. Kleene's Theorem
3. Equation-based alternate construction

Examples of Regular Expressions

Expressions built from a, b, ϵ, using operators $+$, $\cdot$, and $*$.

- $(a^* + b^*) \cdot c$
  "Strings of only a's or only b's, followed by a c."
- $(a+b)^* abb (a+b)^*$
  "contains abb as a subword."
- $(a+b)^* b (a+b)(a+b)$
  "3rd last letter is a b."
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  $(0000 + 0011 + \cdots + 1111)^*(\epsilon + 0 + 1 + \cdots + 111)$

Formal definitions

- Syntax of regular expresions over an alphabet A:

$$r ::= \emptyset \mid a \mid r + r \mid r \cdot r \mid r^*$$

  where $a \in A$.

- Semantics: associate a language $L(r) \subseteq A^*$ with regexp r.

$$\begin{array}{lcl} L(\emptyset) & = & \{\} \\ L(a) & = & \{a\} \\ L(r + r') & = & L(r) \cup L(r') \\ L(r \cdot r') & = & L(r) \cdot L(r') \\ L(r^*) & = & L(r)^*. \end{array}$$

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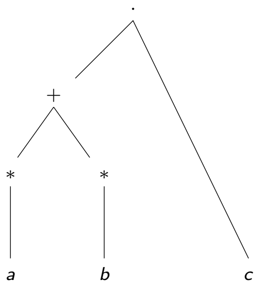
- Question: Do we need ϵ in syntax?
  No. $\epsilon \equiv \emptyset^*$.

Example: Semantics of regexp

$(a^* + b^*) \cdot c$



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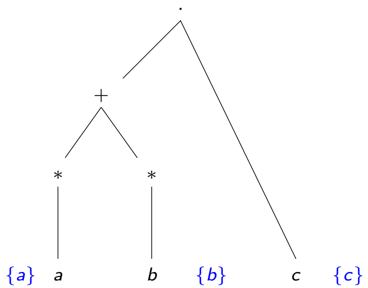

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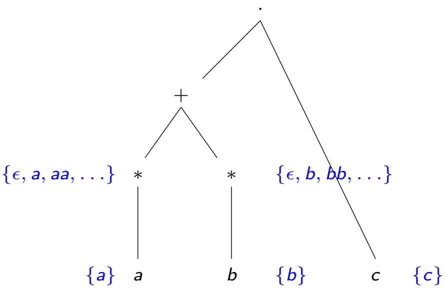

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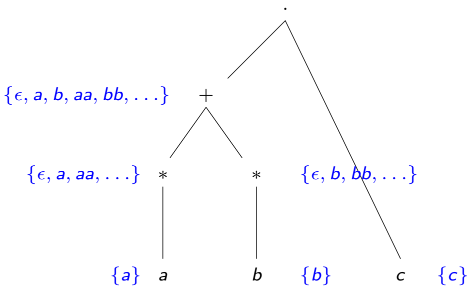

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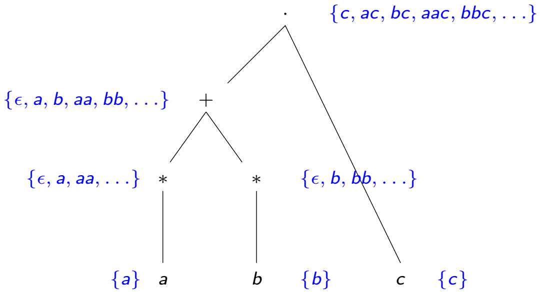


Kleene's Theorem: RE = DFA

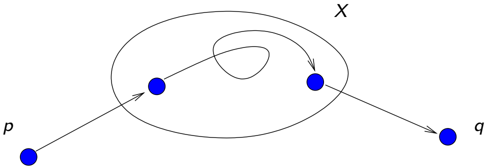
Class of languages defined by regular expressions coincides with regular languages.

Proof

- RE $\rightarrow$ DFA: Use closure properties of regular languages.
- DFA $\rightarrow$ RE:

DFA $\longrightarrow$ RE: Kleene's construction

- Let $\mathcal{A} = (Q, s, \delta, F)$ be given DFA.
- Define $L_{pq} = \{w \in A^* \mid \widehat{\delta}(p, w) = q\}$.
- Then $L(\mathcal{A}) = \bigcup_{f \in F} L_{sf}$.
- For $X \subseteq Q$, define $L^X_{pq} = \{w \in A^* \mid \widehat{\delta}(p, w) = q \text{ via a path that stays in } X \text{ except for first and last states}\}$



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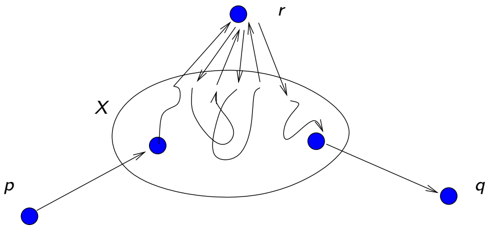
DFA $\longrightarrow$ RE: Kleene's construction



Advantage:

$$L_{pq}^{X \cup \{r\}} = L_{pq}^{X} + L_{pr}^{X} \cdot (L_{rr}^{X})^{*} \cdot L_{rq}^{X}.$$

DFA $\longrightarrow$ RE: Kleene's construction (2)

Method:

- Begin with L_{sf}^{Q} for each $f \in F$.
- Simplify by using terms with strictly smaller X's:

$$L_{pq}^{X \cup \{r\}} = L_{pq}^{X} + L_{pr}^{X} \cdot (L_{rr}^{X})^{*} \cdot L_{rq}^{X}.$$

- For base terms, observe that

$$L_{pq}^{\{\}} = \begin{cases} \{a \mid \delta(p,a) = q\} & \text{if} \quad p \neq q \\ \{a \mid \delta(p,a) = q\} \cup \{\epsilon\} & \text{if} \quad p = q. \end{cases}$$

- Exercise: convert NFA/DFA's below to RE's:

DFA → RE: Kleene's construction (2)

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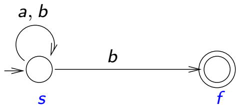
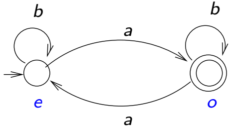


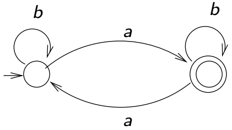


DFA → RE using system of equations

- Aim: to construct a regexp for

$$L_q = \{w \in A^* \mid \widehat{\delta}(q, w) \in F\}.$$

- Note that $L(\mathcal{A}) = L_s$.
- Example:



Set up equations to capture L_q's:

$$\begin{aligned} x_e &= b \cdot x_e + a \cdot x_o \\ x_o &= a \cdot x_e + b \cdot x_o + \epsilon. \end{aligned}$$

- Solution is a RE for each x, such that languages denoted by LHS and RHS RE's coincide.

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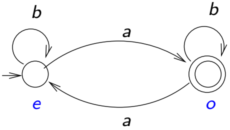

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Solutions to a system of equations

- L_q's are a solution to the system of equations
- In general there could be many solutions to equations.
  - Consider $x = A^*x$ (Here A is the alphabet). What are the solutions to this equation?
- In the case of equations arising out of automata, L_q's can be seen to be the unique solution to the equations.

Computing the least solution to a system of equations

- Equations arising from our automaton can be viewed as:

$$\begin{bmatrix} x_e \\ x_o \end{bmatrix} = \begin{bmatrix} b & a \\ a & b \end{bmatrix} \begin{bmatrix} x_e \\ x_o \end{bmatrix} + \begin{bmatrix} \epsilon \\ \emptyset \end{bmatrix}$$

- System of linear equations over regular expressions have the general form:

$$X = AX + B$$

  where X is a column vector of n variables, A is an nxn matrix of regular expressions, and B is a column vector of n regular expressions.
- Claim: The column vector A^*B represents the least solution to the equations above. [See Kozen, Supplementary Lecture A].
- Definition of A^* when A is a 2x2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} (a+bd^*c)^* & (a+bd^*c)^*bd^* \\ (d+ca^*b)^*ca^* & (d+ca^*b)^* \end{bmatrix}$$