Büchi Automata and their closure properties

- Ajith S and Ankit Kumar

Motivation

- Conventional programs accept input, compute, output result, then terminate
- *Reactive program* : not expected to terminate
- “react” to their environment continuously
- eg: discrete event controllers, schedulers, operating systems, network protocols, infinite computation involving multiple agents etc...

Modelling infinite computations of reactive systems

- An infinite string (ω word) α over Σ is a mapping $\mathbb{N} \rightarrow \Sigma$ and is shown by the infinite sequence

  $$a_0, a_1, a_2 \ldots.$$

- Set of all ω words over Σ is denoted by

  $\Sigma^\omega = \{a_1 a_2 a_3 \ldots \mid a_i \in \Sigma\}$

- A subset $L \subseteq \Sigma^\omega$ is called ω-language

Examples

Let $\Sigma = \{a,b\}$, $L = \{\alpha \mid \alpha \in \Sigma^{\omega}$, α has infinitely many b's$\}$

Then L is an ω language and $\alpha = a^{*}b^{\omega} \in L$, $\beta = (ab)^{\omega} \in L$ and $\gamma = (ba)^{\omega} \in L$.

Büchi Automata - Formal Definition

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

By Σ^{ω} we denote the set of all infinite words over Σ .

A *non-deterministic Büchi automaton* (NBA) over Σ is a tuple $A = \langle Q, I, T, F \rangle$, where:

- Q is a finite set of *states*,
- $I \subseteq Q$ is a set of *initial states*,
- $T \subseteq Q \times \Sigma \times Q$ is a *transition relation*,
- $F \subseteq Q$ is a set of *final states*.

Acceptance Condition

A *run* of a Büchi automaton is defined over an infinite word $\omega : \alpha_1\alpha_2 \ldots$ as an infinite sequence of states $\pi : q_0q_1q_2 \ldots$ such that:

- $q_0 \in I$ and
- $(q_i, \alpha_{i+1}, q_{i+1}) \in \mathrm{T}$, for all $i \in \mathbb{N}$.

Let $\mathrm{inf}(\pi) = \{\mathrm{q} \mid \mathrm{q}$ appears infinitely often on $\pi\}$.

Run π of A is said to be *accepting* iff $inf(\pi) \cap F \neq \phi$.

Example

The following finite ω - automata with $F = \{q_1, q_2\}$ accept the ω - language

$$A = \{ \alpha \in \{a,b\}^\omega \mid \alpha \text{ ends with either } a^\omega \text{ or } (ab)^\omega \}$$

$$= \sum^* a^\omega \cup \sum^* (ab)^\omega$$

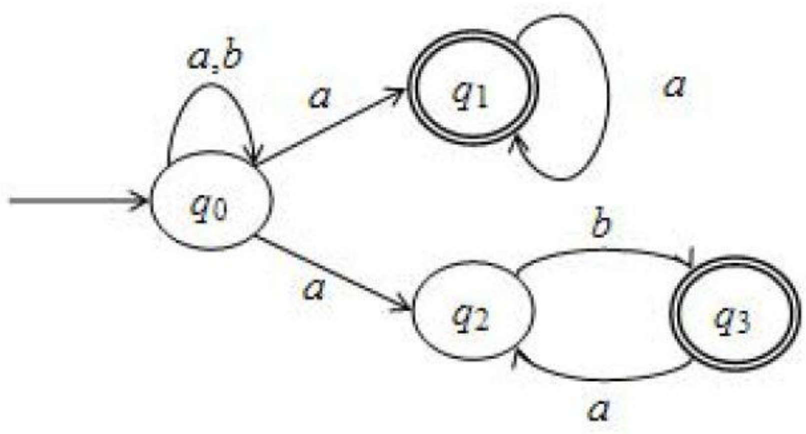


Figure 2: Finite ω-automata which accept A

Closure Properties

Whether a Büchi automaton is closed under the following operations ??

- *union*
- *intersection*
- *concatenation*
- *complementation*

Closure Under Union

Lemma : If L_1 and L_2, languages over $\sum^{\omega}$, are recognised by Büchi automata $B_1 = (Q_1, I_1, T_1, F_1)$ and $B_2 = (Q_2, I_2, T_2, F_2)$ respectively, then $L_1 \cup L_2$ is also recognised by a Büchi automaton.

Proof : $w.l.o.g$ we assume that $Q_1 \cap Q_2 = \phi$

- Recogniser for L is $B = (Q_1 \cup Q_2, I_1 \cup I_2, T_1 \cup T_2, F_1 \cup F_2)$.

- **Claim** : Any $\alpha \in L_1 \cup L_2$ is accepted by B

If α has an successful run on B_1, then it also has a successful run on B. Similar is the case if has a successful run on B_2.In the other direction, if has a successful run on B, then it must have a successful run on B_1 or B_2 component of B.

By our construction these two components are disjoint.

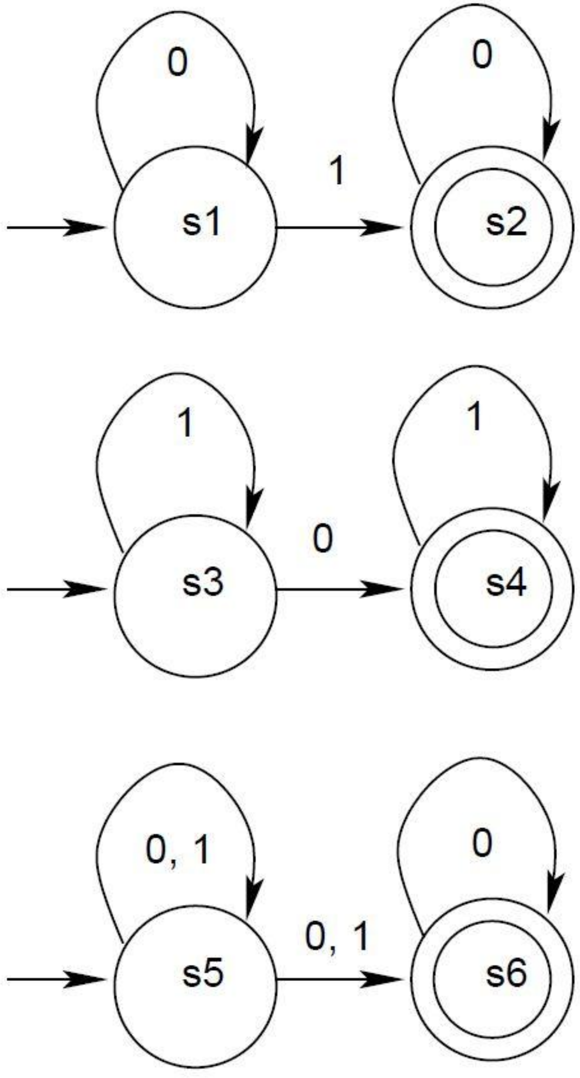
Example

Consider the following two languages:

L_1: each string has either exactly one 0 or exactly one 1.

L_2: there are finitely many 1's in each sequence.

The automaton for $L = L_1 \cup L_2$ is



Closure Under Intersection

Lemma : If L_1 and L_2, languages over $\sum^{\omega}$, are recognised by Büchi automata $B_1 = (Q_1, I_1, F_1, T_1)$ and $B_2 = (Q_2, I_2, F_2, T_2)$ respectively, then $L_1 \cap L_2$ is also recognised by a Büchi automaton.

Proof : using product construction

- Recogniser for L is $B = (Q, I, T, F)$ where

Q - $Q_1 X Q_2 X \{1,2\}$

I - $I_1 X I_2 X \{1\}$

T - $\Delta_1 \cup \Delta_2$

$\Delta_1 = \{((q_1,q_2,1), a, (q'_1,q'_2,i)) \mid (q_1,a,q'_1) \in \Delta_1$ and $(q_2,a,q'_2) \in \Delta_2$ and if $q_1 \in F_1$ then $i=2$ else $i=1$ $\}$

$\Delta_2 = \{((q_1,q_2,2), a, (q'_1,q'_2,i)) \mid (q_1,a,q'_1) \in \Delta_1$ and $(q_2,a,q'_2) \in \Delta_2$ and if $q_2 \in F_2$ then $i=1$ else $i=2$ $\}$

F - $\{ (Q_1 , Q_2 , 2) \mid Q_2 \in F_2 \}$

Claim : By construction

$r' = (q^0{}_1, q^0{}_2, i^0), (q^1{}_1, q^1{}_2, i^1),\ldots$ is a run of automaton B on input word ω iff $r_1 = q^0{}_1, q^1{}_1,\ldots$ is run of B_1 on ω and $r_2 = q^0{}_2, q^1{}_2,\ldots,\ldots$ is run of B_2 on ω.

r_1 is accepting and r_2 is accepting iff r' is concatenation of an infinite series of finite segments of 1-states and 2-states alternatively.

There is such a series of segments of r' iff r' is accepted by B.

Closure Under Concatenation

Lemma : If L_1 and L_2, languages over $\sum^\omega$, are recognised by Büchi automata $B_1 = (Q_1, I_1, T_1, F_1)$ and $B_2 = (Q_2, I_2, T_2, F_2)$ respectively, then $L_1 . L_2$ is also recognised by a Büchi automaton.

Proof : $w.l.o.g$ we assume that $Q_1 \cap Q_2 = \phi$

- Recogniser for L is $B = (Q_1 \cup Q_2, \; I, \; T, \; F_1 \cup F_2)$.

  $T = T_1 \cup T_2 \cup \{ (q,a,q') \mid q' \in I_2$ and $\exists f \in F_1 . (q,a,f) \in T_1 \}$

  if $I_1 \cap F_1$ is empty then $I = I_1$ otherwise $I = I_1 \cup I_2$

Buildup for Complementation

Buchi Characterization

Lemma 1 : If a regular language is accepted by a DFA $M = \{Q,s,\delta,F\}$ over $\sum$, there is an equivalent DFA $M' = \{Q',s,\delta',F'\}$ with no incoming transitions to start state.

Lemma 2 : If language $L \subseteq \sum^*$ is recognized by M, then there is a Buchi automaton for L^ω

Lemma 3 : If L_1,L_2 are regular, then $L_1L_2^\omega$ is Buchi recognizable.

Buildup for Complementation

Theorem 1: An ω language is Buchi recognizable iff $L = L_i K_i^{\omega}$ where L_i , $K_i \subseteq \Sigma^*$ are regular.

Corollary : Any non-empty Buchi recognizable language contains an ultimate periodic word of the form uvvv…

Buildup for Complementation

Equivalence Classes: Define a binary relation $\sim_B$ on Σ^*.

Let $x,y \in \Sigma^*$, then $x \sim_B y$ iff

i) $\forall (p,q) \in Q$, $p \rightarrow_x q$ iff $p \rightarrow_y q$

ii) $\forall (p,q) \in Q$, $p \rightarrow_x^F q$ iff $p \rightarrow_y^F q$

Lemma 4: $\sim_B$ is an equivalence relation (reflexive, symmetric and transitive)

Lemma 5: $\sim_B$ is a congruence relation (due to transitivity)

Buildup for Complementation

Lemma 6: $\sim_B$ has finite index i.e. partitions Σ^* in finite number of equivalence classes.

Corollary : From Myhill-Nerode theorem, if there is an equivalence relation of finite index over Σ^*, and the relation is a congruence relation, then the equivalence classes are regular languages.

Lemma 7: Given a Buchi automaton B = $\{Q,s,\delta,F\}$, the equivalence class $[x]_{\sim B}$ is the intersection of all languages of the form $L_{p,q}$, $L_{p,q}^F$, $\Sigma^*\backslash L_{p,q}$ and $\Sigma^*\backslash L_{p,q}^F$.

Complementation

Approach to show closure under complementation :

- we have already shown construction of a congruence relation $\sim_B$
- $\sim_B$ divides Σ^* into finite number of equivalence classes, each of which is regular.
- For any 2 equivalence classes L_1 and L_2 , $L_1L_2^\omega$ is Buchi recognizable.
- Given a Buchi automaton B, we will show that if $L_1L_2^\omega \cap L_\omega(B) \neq \Phi$ then $L_1L_2^\omega \subseteq L_\omega(B)$.
- We also need to show that each $\alpha \in L_\omega(B)$ belongs to some $L_1L_2^\omega$ where L_1 and L_2 are equivalence classes.

Buildup for Complementation

Lemma 8 : $\sim_B$ defined on Σ^* saturates L if for any pair of equivalence classes L_1 and L_2, whenever $L_1L_2^\omega \cap L_\omega \neq \Phi$ then $L_1L_2^\omega \subseteq L$

Lemma 9 : Given a Buchi automata B = (Q,s,δ,F) and an ω-string α, we define a binary relation on the positions in the string, or on the set of positive integers $\mathbb{N}$ as follows:

$k \sim_\alpha l$, if there is a position $m > k, l$, such that $\alpha_{k,m} \sim_\alpha \alpha_{l,m}$

Clearly $\sim_\alpha$ is reflexive and symmetric.

It is also transitive.

Buildup for Complementation

Lemma 10 : Given a Buchi automata B = (Q,s,δ,F) and an ω-string α, the equivalence relation $\sim_\alpha$ has finite index i.e. it divides $\mathbb{N}$ in finite number of equivalence classes.

Corollary : As there are infinite number of letters in an ω-string α, but only finitely many equivalence classes, there must be an equivalence class with infinite number of positions in it.

We define two subsets of Σ^* as follows:

$L_1 = \{x \in \Sigma^* : x \sim_B \alpha_{1,p0}\}$, $L_2 = \{x \in \Sigma^* : x \sim_B \alpha_{p0,p1}\}$.

Clearly L_1 and L_2 are two equivalence classes under $\sim_B$.

Closure Under Complementation

Lemma 11 : The ω-string $\alpha \in L_1 L_2^\omega$

Lemma 12 : Let B be a Buchi automaton over Σ. The language $\Sigma^\omega \setminus L_\omega(B)$ is Buchi recognizable.

In the previous theorem we proved that any ω-string is a member of some $L_1 L_2^\omega$. So, the complement language, $\Sigma^\omega \setminus L_\omega(B) = \cup L_1 L_2^\omega$, where L_1 and L_2 are equivalence classes generated by B, and the union is over all pairs of such L_1 and L_2 such that $L_1 L_2^\omega \cap L_\omega(B) = \Phi$.
So Buchi recognisable languages are closed under complementation.

References

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