

Undecidable problems about CFL's

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Outline

- 1 Some Decidable/Undecidable problems about CFL's

Problems about CFL's

Problem (a)

Is it decidable whether a given CFG accepts a non-empty language?

Problems about CFL's

Problem (a)

Is it decidable whether a given CFG accepts a non-empty language?

Yes, it is. We can find out which non-terminals of G can derive a terminal string: i.e. there exists a derivation $X \xRightarrow{*} w$ for some terminal string w .

- Maintain a set of “marked” non-terminals. Initially $N_{marked} = \emptyset$.
- Mark all non-terminals X such that $X \rightarrow w$ is a production in G .
- Repeat until we are unable to mark any more non-terminals:
 - Mark X if there exists a production $X \rightarrow \alpha$ such that $\alpha \in (A \cup N_{marked})^*$.
- Return “Non-empty” if $S \in N_{marked}$, else return “Empty.”

Problems about CFL's

Problem (b)

Is it decidable whether a given CFG accepts a finite language?

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Problem (b)

Is it decidable whether a given CFG accepts a finite language?

Yes, it is.

- Convert G to CNF.
- Check if there is a parse tree of depth $n + 1$ where n is the number of non-terminals. $L(G)$ is infinite iff there is a parse tree of depth $n + 1$ or more.

Problems about CFL's

Problem (c)

Is it decidable whether a given CFG G is **universal**. That is, is $L(G) = A^*$?

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No, it is undecidable (not even r.e.).

Undecidability of universality of a CFL

- We can reduce \neg HP to the problem of universality of a CFG:

$$\neg\text{HP} \leq \text{Universality of CFG.}$$

- Given a TM M and input x , we can construct a CFG $G_{M,x}$ over an input alphabet Δ such that

M does not halt on x iff $G_{M,x}$ is universal (i.e. $L(G_{M,x}) = \Delta^$).*

- Hence the problem is non-r.e.

Encoding computations of M on x

Let $M = (Q, A, \Gamma, s, \delta, \vdash, \dashv, t, r)$ be a given TM and let $x = a_1 a_2 \cdots a_n$ be an input to it.

We can represent a configuration of M as follows:

$$\begin{array}{cccccc} \vdash & b_1 & b_2 & b_3 & \cdots & b_m \\ - & - & q & - & & - \end{array}$$

Thus a configuration is encoded over the alphabet $\Gamma \times (Q \cup \{-\})$.

Encoding computations of M on x

A computation of M on x is a string of the form

$$c_0 \# c_1 \# \cdots \# c_N \#$$

such that

- ① Each c_i is the encoding of a configuration of M .
- ② c_0 is (encoding of) the start configuration of M on x .

$$\begin{array}{ccccccc} \vdash & a_1 & a_2 & a_3 & \cdots & a_n & \\ s & - & - & - & & - & \end{array}$$

- ③ All c_i 's are of **same** length, and maximal (in at least one config the head is at the last position).
- ④ Each $c_i \xrightarrow{1} c_{i+1}$, and
- ⑤ c_N is a halting configuration (i.e. state component is t or r).



Describing $Valcomp_{M,x}$

The language $Valcomp_{M,x}$ over the alphabet

$$\Delta = \Gamma \times (Q \cup \{-\}) \cup \{\#\}$$

can be described as the intersection of

- $L_1 \subseteq (C \cdot \#)^*$ where C is the set of valid encodings of configurations of M , beginning with initial config, and containing one config with a t or r state.
- L_2 which makes sure each c_i is of the same length.
- $L_3 = \{c_0\# \cdots \#c_N\# \mid N \geq 1, c_i \xrightarrow{1} c_{i+1}\}$.

Hence $\neg Valcomp_{M,x} = \overline{L_1} \cup \overline{L_2} \cup \overline{L_3}$.

Claim

$\neg Valcomp_{M,x}$ is a CFL (in fact *regular*) and given M and x , we can construct a PDA/CFG $G_{M,x}$ that accepts it.

Proof of claim

Claim

Given M, x , we can construct a PDA/CFG $G_{M,x}$ for $\neg Valcomp_{M,x}$.

- We know $\neg Valcomp_{M,x} = \overline{L_1} \cup \overline{L_2} \cup \overline{L_3}$.
- L_1 is regular, and $\overline{L_2}$ is a CFL ($L_2 = L_2^o \cap L_2^e$, and each is DCFL).
- $\overline{L_3}$ is a CFL
 - Claim: $c \stackrel{1}{\Rightarrow} d$ iff at every position i the 3 symbols $c(i), c(i+1), c(i+2)$ in c and $d(i), d(i+1), d(i+2)$ in d , are "valid" pairs of triples.
 - Example: if $(s, \vdash), (p, \vdash, R)$ is a move of M then foll pair of triples is valid:

$$\left\langle \begin{array}{ccc} \vdash & a_1 & a_2 \\ s & - & - \end{array} , \begin{array}{ccc} \vdash & a_1 & a_2 \\ - & p & - \end{array} \right\rangle$$

- So is

$$\left\langle \begin{array}{ccc} a & b & c \\ - & - & - \end{array} , \begin{array}{ccc} a & b & c \\ - & - & - \end{array} \right\rangle$$

Proof of claim

- Example: if $(p, a) \rightarrow (q, b, R)$ is a move of M then foll is **invalid**:

$$\left\langle \begin{array}{ccc} a & b & c \\ p & - & - \end{array}, \begin{array}{ccc} b & b & c \\ - & - & - \end{array} \right\rangle$$

- So is

$$\left\langle \begin{array}{ccc} a & b & c \\ - & - & - \end{array}, \begin{array}{ccc} a & b & c \\ - & - & - \end{array} \right\rangle$$

- Thus there is a finite table of valid triples that we can compute based on M .
- Now use a (non-det) PDA to guess a config c_k and a position i in it, and accept if the triple at $c_k(i)$ and $c_{k+1}(i)$ are **not** valid.
- So $\overline{L_3}$ is a CFL.
- Construct a PDA/CFG $G_{M,x}$ that accepts the union of $\overline{L_1}$, $\overline{L_2}$, and $\overline{L_3}$.

Problems about CFL's

Problem (d)

Is it decidable whether the intersection of two given CFG's is non-empty?

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No, it is undecidable. Given M and x , describe 2 PDA's that accept computations of the form:



Here each shaded configuration is in **reversed** form.

- PDA M_1 checks that each even-numbered configuration is correctly followed by the next configuration.
- PDA M_2 checks that each odd-numbered configuration is correctly followed by the next configuration.
- In fact, a **DPDA** can check correct consecution of consecutive even-odd (respectively odd-even) configurations.

Other undecidable problems about CFL's

Problem (e)

Is it decidable whether the intersection of two given CFL's is a CFL?

Problem (f)

Is it decidable whether the complement of a given CFL is a CFL?

Problem (g)

Is it decidable whether a given CFL is a DCFL?

Other undecidable problems about CFL's

Problem (e)

Is it decidable whether the intersection of two given CFL's is a CFL?

Problem (f)

Is it decidable whether the complement of a given CFL is a CFL?

Problem (g)

Is it decidable whether a given CFL is a DCFL?

All undecidable. Exercise!