

Pushdown Automata

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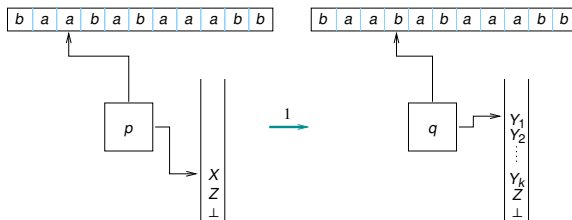
Outline

- 1 Pushdown Automata
- 2 Definitions
- 3 Exercise
- 4 Equivalence of acceptance by FS and ES

Pushdown Automata + CFG: history

- CFG's were introduced by Noam Chomsky in 1956.
- Oettinger introduced PDA's for parsing applications in 1961.
- Chomsky, Schutzenberger, and Evey showed equivalence of CFG's and PDA's in 1962.

How a PDA works



Each step of the PDA looks like:

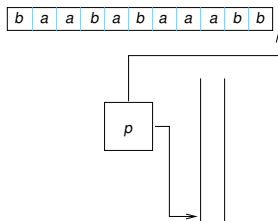
- Read current symbol and advance head;
- Read and pop top-of-stack symbol;
- Push in a string of symbols on the stack;
- Change state.

Each transition Looks like

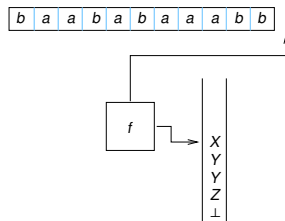
$$(p, a, X) \rightarrow (q, Y_1 Y_2 \cdots Y_k).$$

Two mechanisms of acceptance

Mechanism used must be specified a priori in the PDA definition.



Empty stack



Final State

Accept input if

- Input is consumed and stack is empty (Acceptance by **Empty Stack**).
- Or, input is consumed and PDA is in a final state (Acceptance by **Final State**).

Example PDA

Example PDA (acceptance by empty stack) for $\{a^n b^n \mid n \geq 0\}$

$$\begin{aligned}(s, \epsilon, \perp) &\rightarrow (s, \epsilon) \\(s, a, \perp) &\rightarrow (p, A) \\(p, a, A) &\rightarrow (p, AA) \\(p, b, A) &\rightarrow (q, \epsilon). \\(q, b, A) &\rightarrow (q, \epsilon).\end{aligned}$$

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Illustrate run on input “aaabbb”.

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Illustrate run on input “aaabbb”.

What happens on input “aaabbbb”?

PDA's more formally

A Pushdown Automaton is a structure of the form

$$\mathcal{M} = (Q, A, \Gamma, s, \delta, \perp, F)$$

where

- Q is a finite set of states,
- A is the input alphabet,
- Γ is the stack alphabet,
- $s \in Q$ is the start state,
- $\delta \subseteq_{fin} Q \times (A \cup \{\epsilon\}) \times \Gamma \times Q \times \Gamma^*$ is the (non-deterministic) transition relation,
- $\perp \in \Gamma$ is the bottom-of-stack symbol,
- $F \subseteq Q$ is the set of final states.

Configurations, runs, etc. of a PDA

- A **configuration** of $\mathcal{M}$ is of the form $(p, u, \gamma) \in Q \times A^* \times \Gamma^*$, which says “ $\mathcal{A}$ is in state p , with unread input u , and stack contents γ ”.
- Initial configuration of $\mathcal{M}$ on input w is $(s, w, \perp)$.
- 1-step transition of $\mathcal{M}$: If $(p, a, X) \rightarrow (q, \alpha)$ is a transition in δ , then

$$(p, au, X\beta) \xRightarrow{1} (q, u, \alpha\beta).$$

- Similarly, if $(p, \epsilon, X) \rightarrow (q, \alpha)$ is a transition in δ , then

$$(p, u, X\beta) \xRightarrow{1} (q, u, \alpha\beta).$$

- $\mathcal{M}$ accepts w by empty stack if $(s, w, \perp) \xRightarrow{*} (q, \epsilon, \epsilon)$.
- $\mathcal{M}$ accepts w by final state if $(s, w, \perp) \xRightarrow{*} (f, \epsilon, \gamma)$ for some $f \in F$ and $\gamma \in \Gamma^*$.
- Language accepted by $\mathcal{M}$ is denoted $L(\mathcal{M})$.

Exercise

Design PDA's for the following languages:

- Balanced Parenthesis
- $\{a, b\}^* - \{ww \mid w \in \{a, b\}^*\}$.

Solution

PDA (acceptance by empty stack) for BP

$$(s, \epsilon, \perp) \rightarrow (s, \epsilon)$$

$$(s, (, \perp) \rightarrow (s, A\perp)$$

$$(s, (, A) \rightarrow (s, AA)$$

$$(s,), A) \rightarrow (s, \epsilon).$$

Equivalence of acceptance criteria

Claim

- Given a PDA $\mathcal{M}$ that accepts by Final State we can give a PDA $\mathcal{M}'$ that accepts by Empty Stack such that $L(\mathcal{M}') = L(\mathcal{M})$.
- Conversely, given a PDA $\mathcal{M}$ that accepts by Empty Stack we can give a PDA $\mathcal{M}'$ that accepts by Final State such that $L(\mathcal{M}') = L(\mathcal{M})$.

In fact given a PDA $\mathcal{M}$ we can construct a PDA $\mathcal{M}'$ that accepts the same language as $\mathcal{M}$, by **both** acceptance criteria.

From Final State to ES/FS

What is the problem in doing this?

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- $\mathcal{M}$ may reject an input by emptying its stack.

From Final State to ES/FS

What is the problem in doing this?

- $\mathcal{M}$ may reject an input by emptying its stack.
- Let $\mathcal{M} = (Q, A, \Gamma, s, \delta, \perp, F)$.
- Define $\mathcal{M}' = (Q \cup \{s', t\}, A, \Gamma \cup \{\perp\}, s', \delta', \perp, \{t\})$, where δ' is δ plus the transitions:

$$\begin{array}{lll}
 (s', \epsilon, \perp) & \rightarrow & (s, \perp \perp) \\
 (s, a, \perp) & \rightarrow & (p, A) \quad \text{original transition in } \delta \\
 (f, \epsilon, X) & \rightarrow & (t, X) \quad \text{for } X \in \Gamma \cup \{\perp\} \\
 (t, \epsilon, X) & \rightarrow & (t, \epsilon) \quad \text{for } X \in \Gamma \cup \{\perp\}.
 \end{array}$$

- Argue that if $w \in L(\mathcal{M})$ then $w \in L(\mathcal{M}')$.
- Argue that if $w \in L(\mathcal{M}')$ then $w \in L(\mathcal{M})$.

From Empty Stack to ES/FS

- Let $\mathcal{M} = (Q, A, \Gamma, s, \delta, \perp)$.
- Define $\mathcal{M}' = (Q \cup \{s', t\}, A, \Gamma \cup \{\perp\}, s', \delta', \perp, \{t\})$, where δ' is δ plus the transitions:

$$\begin{aligned}(s', \epsilon, \perp) &\rightarrow (s, \perp \perp) \\(q, \epsilon, \perp) &\rightarrow (t, \perp) \quad \text{for } q \in Q \\(t, \epsilon, \perp) &\rightarrow (t, \epsilon).\end{aligned}$$

- Argue that if $w \in L(\mathcal{M})$ then $w \in L(\mathcal{M}')$.
- Argue that if $w \in L(\mathcal{M}')$ then $w \in L(\mathcal{M})$.