Myhill-Nerode Theorem

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Outline

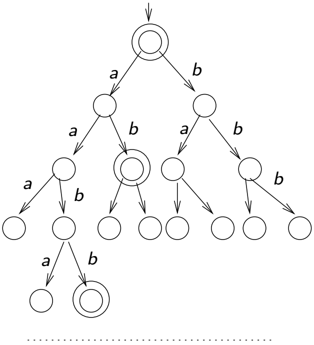
Myhill-Nerode Theorem: Overview

- Every language L has a "canonical" deterministic automaton accepting it.
  - Every other DA for L is a "refinement" of this canonical DA.
  - There is a unique DA for L with the minimal number of states.
- Holds for any L (not just regular L).
- L is regular iff this canonical DA has a finite number of states.
- There is an algorithm to compute this canonical DA from any given finite-state DA for L.

DA for any language

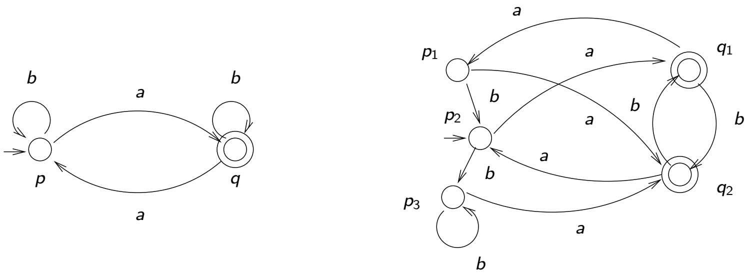
Note that every language L has DA accepting it (we call this the "free" DA for L).

The free DA for $L = \{a^n b^n \mid n \geq 0\}$:



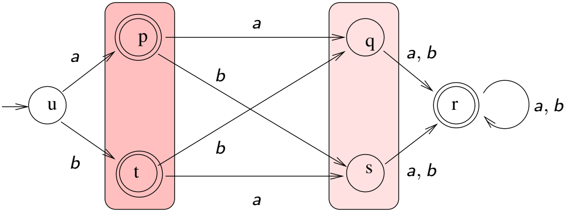
Illustrating "refinement" of DA: Example 0

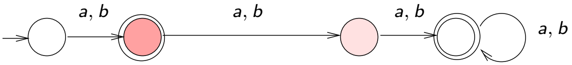
- Replicate each state in the first automaton some number of times, and add an edge labelled a from p_i (a copy of state p) to q_j provided $\delta(p, a) = q$. Then "split" DFA accepts the same language.
- Conversely, every DA for L is a "splitting" of the canonical DA for L.



Illustrating "refinement" of DA: Example 1

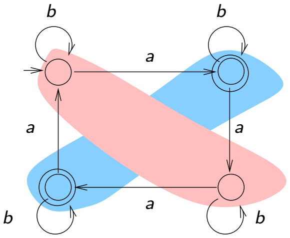
Every DA for L is a "refinement" of this canonical DA:

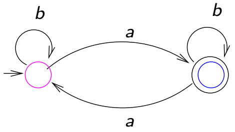




Illustrating "refinement" of DA: Example 2

Every DA for L is a "refinement" of this canonical DA:



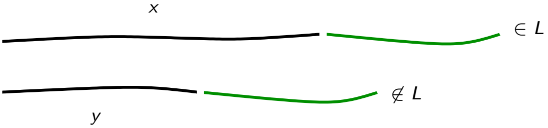


Myhill-Nerode Theorem

Canonical equivalence relation $\equiv_L$ on A^* induced by $L \subseteq A^*$:

$$x \equiv_L y \text{ iff } \forall z \in A^*,\ xz \in L \text{ iff } yz \in L.$$

$x \not\equiv_L y$ iff



Theorem (Myhill-Nerode)

L is regular iff $\equiv_L$ is of finite index (that is has a finite number of equivalence classes).

Exercise 1

Describe the equivalence classes for $L =$ "Odd number of a's".

Exercise 2

Describe precisely the equivalence classes of $\equiv_L$ for the language $L \subseteq \{a, b\}^*$ comprising strings in which the 2nd last letter is a b.

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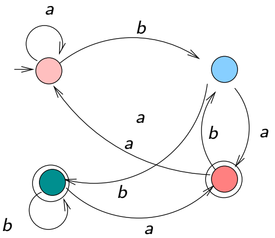

ϵ, a, $.*aa$	b, $.*ab$
$.*bb$	$.*ba$

Exercise 3

Describe the equivalence classes of $\equiv_L$ for the language $L = \{a^n b^n \mid n \geq 0\}$.

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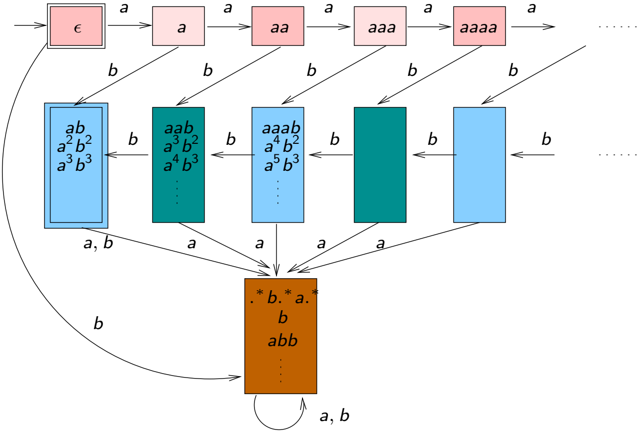

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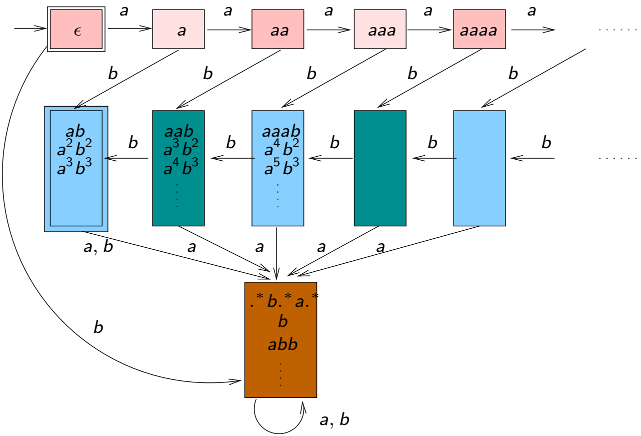
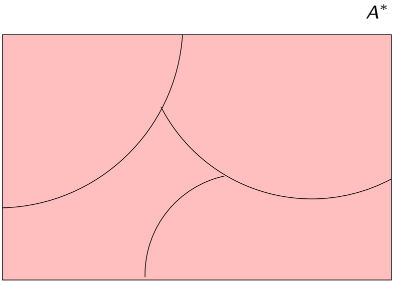


Note: The natural deterministic PDA for L gives this DA.

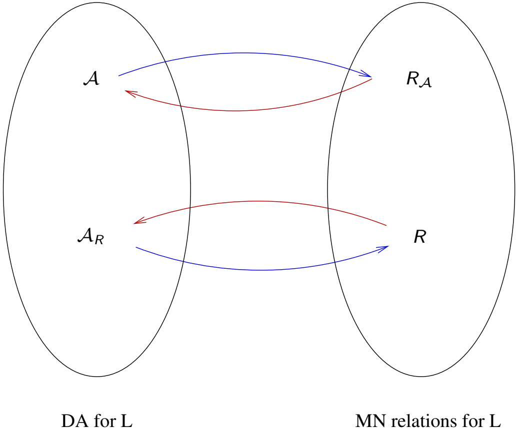
Myhill-Nerode (MN) relations for a language

- An MN relation for a language L on an alphabet A is an equivalence relation R on A^* satisfying
  1. R is right-invariant (i.e. $xRy \implies xaRya$ for each $a \in A$.)
  2. R refines (or "respects") L (i.e. $xRy \implies x, y \in L \text{ or } x, y \notin L$).



Deterministic Automata for L and MN relations for L

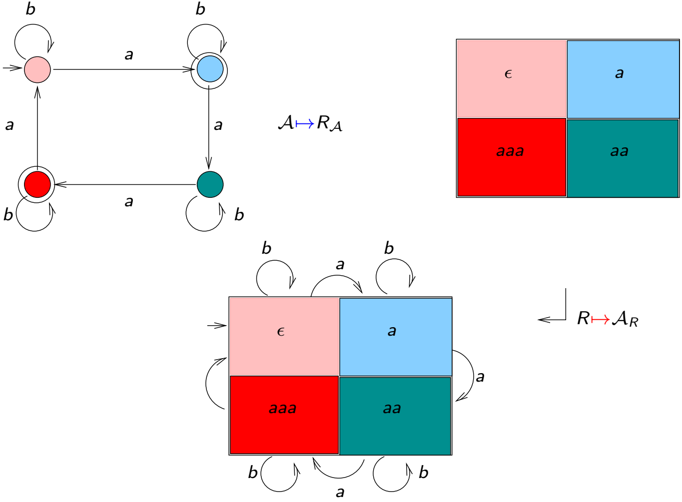
DA for L and MN relations for L are in 1-1 correspondence (they represent eachother).



Maps $\mathcal{A} \mapsto R_{\mathcal{A}}$ and $\mathcal{A}_R \leftarrow R$ are inverses of eachother.

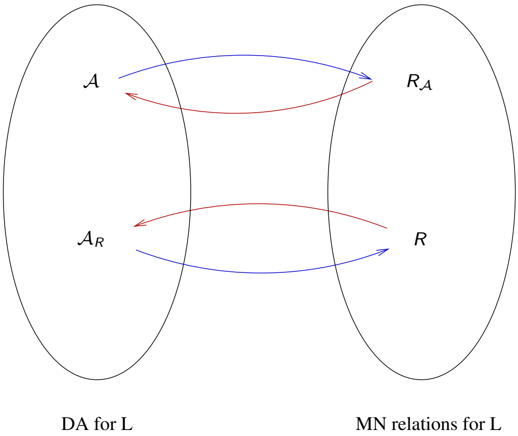
Example DA and its induced MN relation

L is "Odd number of a's":



Deterministic Automata for L and MN relations for L

DA (with no unreachable states) for L and MN relations for L are in 1-1 correspondence.



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Equivalence relations and Refinement

An equivalence relation R on a set X refines another equivalence relation S on X if for each $x, y \in X$, $xRy \implies xSy$.

Exercise: Consider the relations R: "equal mod 2" and S: "equal mod 4". Which refines which? Picture R and S.

Any MN-relation for L refines the relation $\equiv_L$

Lemma

Let L be any language over an alphabet A. Let R be any MN-relation for L. Then R refines $\equiv_L$.

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Proof: To prove that xRy implies $x \equiv_L y$. Suppose $x \not\equiv_L y$. Then there exists z such that (WLOG) $xz \in L$ and $yz \notin L$. Suppose xRy. Since its an MN relation for L, it must be right invariant; and hence $xzRyz$. But this contradicts the assumption that R respects L.

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As a corollary we have:

Theorem (Myhill-Nerode)

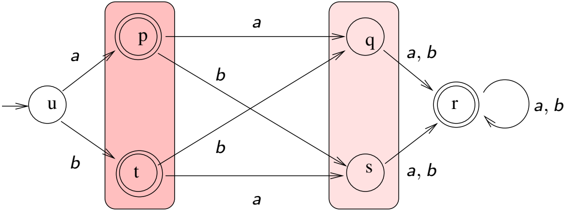
L is regular iff $\equiv_L$ is of finite index (that is has a finite number of equivalence classes).

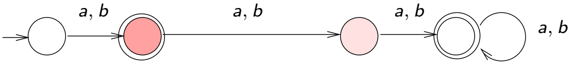
Canonical DA for L

- We call $\mathcal{A}_{\equiv_L}$ the "canonical" DA for L.
- In what sense is $\mathcal{A}_{\equiv_L}$ canonical?
  - Every other DA for L is a refinement of $\mathcal{A}_{\equiv_L}$.
  - $\mathcal{A}$ is a refinement of $\mathcal{B}$ if there is a stable partitioning $\sim$ of $\mathcal{A}$ such that quotient of $\mathcal{A}$ under $\sim$ (written $\mathcal{A}/\sim$) is isomorphic to $\mathcal{B}$.
  - Stable partitioning of $\mathcal{A} = (Q, s, \delta, F)$ is an equivalence relation $\sim$ on Q such that:
    - $p \sim q$ implies $\delta(p, a) \sim \delta(q, a)$.
    - If $p \sim q$ and $p \in F$, then $q \in F$ also.
  - Note that if $\sim$ is a stable partitioning of $\mathcal{A}$, then $\mathcal{A}/\sim$ accepts the same language as $\mathcal{A}$.

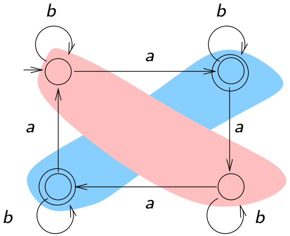
Example: 1

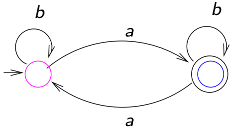
A stable partitioning shown by pink and light pink classes, and below, the quotiented automaton:





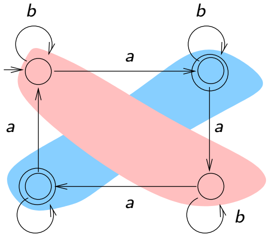
Example: 2



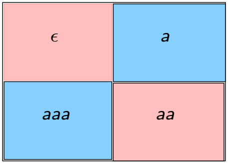


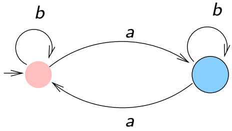
Proving canonicity of $\mathcal{A}_{\equiv_L}$

Let $\mathcal{A}$ be a DA for L with no unreachable states. Then $\mathcal{A}_{\equiv_L}$ represents a stable partitioning of $\mathcal{A}$. (Use the refinement of $\equiv_L$ by the MN relation $R_{\mathcal{A}}$.)

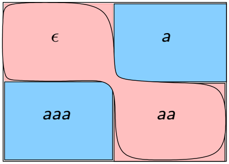


$\mathcal{A} \mapsto R_{\mathcal{A}}$





$\mathcal{A}_{\equiv_L} \leftarrowtail \equiv_L$



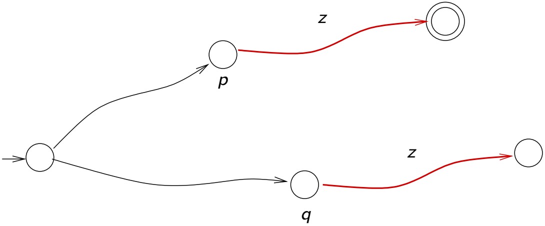
Stable partitioning $\approx$

- Let $\mathcal{A} = (Q, s, \delta, F)$ be a DA for L with no unreach. states.
- The canonical MN relation for L (i.e. $\equiv_L$) induces a "coarsest" stable partitioning $\approx_L$ of $\mathcal{A}$ given by

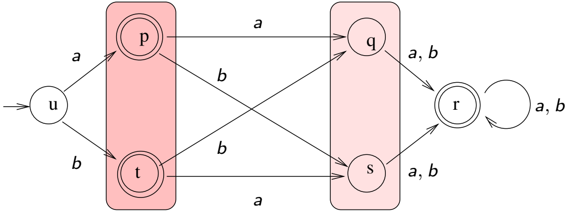
$$p \approx_L q \quad \text{iff} \quad \exists x, y \in A^* \text{ such that } \widehat{\delta}(s, x) = p \text{ and } \widehat{\delta}(s, y) = q, \text{ with } x \equiv_L y.$$

- Define a stable partitioning $\approx$ of $\mathcal{A}$ by

$$p \approx q \text{ iff } \forall z \in A^* : \ \widehat{\delta}(p, z) \in F \text{ iff } \widehat{\delta}(q, z) \in F.$$



Example of $\approx$ partitioning relation



Stable partitioning $\approx$ is coarsest

Claim: $\approx$ coincides with $\approx_L$.

$\approx_L = \approx$.

Proof:

$$p \not\approx q \text{ iff } \exists x, y, z: \ \widehat{\delta}(s, x) = p,\ \widehat{\delta}(s, y) = q, \text{ and } \widehat{\delta}(p, z) \in F \text{ but } \widehat{\delta}(q, z) \notin F.$$

$$\text{iff } p \not\approx_L q.$$

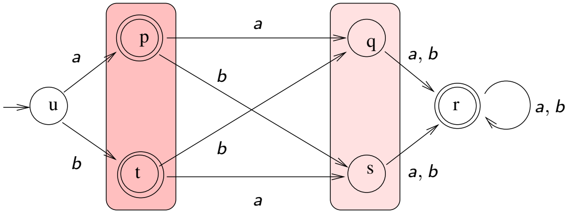
Algorithm to compute $\approx$ for a given DFA

Input: DFA $\mathcal{A} = (Q, s, \delta, F)$.
Output: $\approx$ for $\mathcal{A}$.

1. Initialize entry for each pair in table to "unmarked".
2. Mark (p, q) if $p \in F$ and $q \notin F$ or vice-versa.
3. Scan table entries and repeat till no more marks can be added:
    1. If there exists unmarked (p, q) with $a \in A$ such that $\delta(p, a)$ and $\delta(q, a)$ are marked, then mark (p, q).
4. Return $\approx$ as: $p \approx q$ iff (p, q) is left unmarked in table.

Example

Run minimization algorithm on DFA below:



	u	p	t	q	s	r
u	·					
p		·				
t			·			
q				·		
s					·	
r						·

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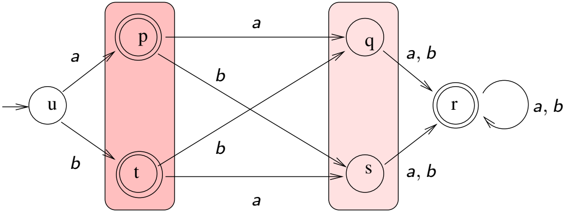

	u	p	t	q	s	r
u	·					
p	√	·				
t	√		·			
q		√	√	·		
s		√	√		·	
r	√			√	√	·

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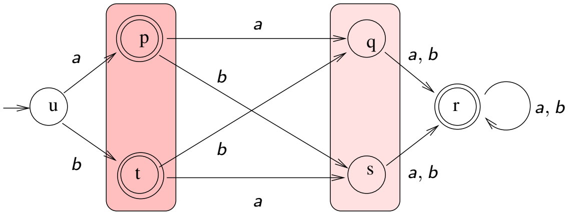

	u	p	t	q	s	r
u	·					
p	√	·				
t	√		·			
q		√	√	·		
s		√	√		·	
r	√	√	√	√	√	·

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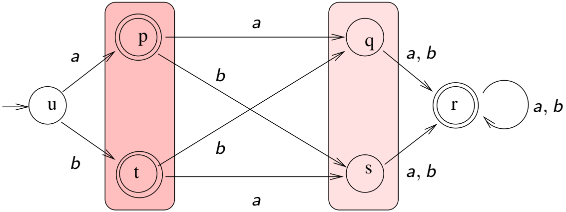

	u	p	t	q	s	r
u	·					
p	√	·				
t	√		·			
q	√	√	√	·		
s	√	√	√		·	
r	√	√	√	√	√	·

Correctness of minimization algorithm

Claim: Algo always terminates.

- $n(n-1)/2$ table entries in each scan, and at most $n(n-1)/2$ scans.
- In fact, number of scans in algo is $\leq n$, where $n = |Q|$.
  1. Consider modified step 3.1 in which mark check is done wrt the table at the end of previous scan.
  2. Argue that at end of i-th scan algo computes $\approx_i$, where

     $$p \approx_i q \text{ iff } \forall w \in A^* \text{ with } |w| \leq i : \ \widehat{\delta}(p, w) \in F \text{ iff } \widehat{\delta}(q, w) \in F.$$

  3. Observe that $\approx_{i+1}$ strictly refines $\approx_i$, unless the algo terminates after scan $i+1$. So modified algo does at most n scans.
  4. Both versions mark the same set of pairs. Also if modified algo marks a pair, original algo has already marked it.

Correctness of minimization algorithm

Claim: Algo marks (p, q) iff $p \not\approx q$.

- $(\Rightarrow)$
- $(\Leftarrow)$