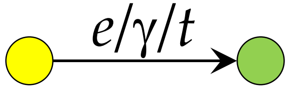
Pushdown Automata

- In a recursive automaton, the stack is implicit
- A PDA is a tuple $P = (Q, A, \Gamma, s, \delta, \bot, F)$ where the stack is explicit
  - Γ is the stack alphabet, $\bot \in \Gamma$ is a special "bottom-of-stack" marker
  - Finite set $\delta \subseteq Q \times (A \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^*$



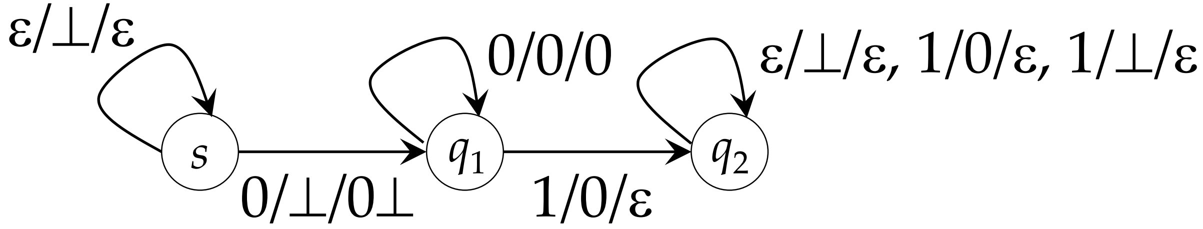
- A configuration c is a pair $(q, t) \in Q \times \Gamma^*$
  - Initial configuration is $(s, \bot)$
  - $(p, \gamma . t_1) \xrightarrow{e} (q, t_2 . t_1)$ if $(p, e, \gamma, q, t_2) \in \delta$ [extend to strings]
- Two notions of PDA acceptance: P accepts w by

Empty Stack (ES): if $(s, \bot) \xrightarrow{w} (q, \varepsilon)$ for some $q \in Q$

Final State (FS): if $(s, \bot) \xrightarrow{w} (q, t)$ for some $q \in F$

Questions

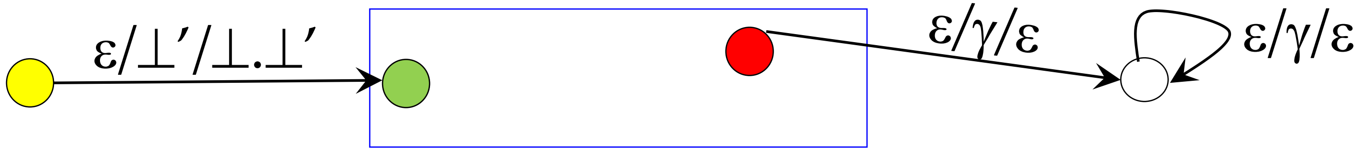
- Your friend claims that the following PDA accepts the language $\{0^n.1^n | n \geq 0\}$ by ES. Show that this claim is false and fix the PDA.



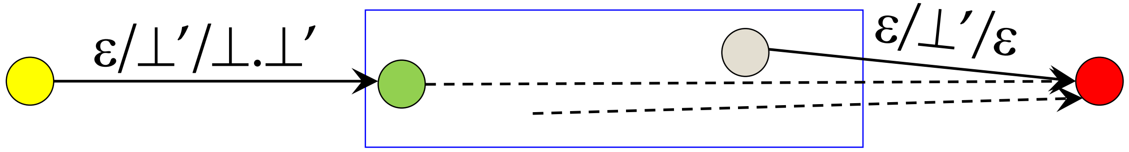
- Modify your PDA to accept the same language by FS.

ES/FS equivalence

- **Claim 1**: For every n-state PDA P that accepts by FS, there is an $(n+2)$-state PDA P' that accepts by ES such that $L(P) = L(P')$.
- **Proof**:



- **Claim 2**: For every n-state PDA P that accepts by ES, there is an $(n+2)$-state PDA P' that accepts by FS such that $L(P) = L(P')$.
- **Proof**:



RA ↔ CFG ↔ PDA

- RA → PDA (by ES and FS):
  - States of PDA = stack alphabet = set of RA states
  - Replace every internal transition (p, e, q) in the RA with the PDA transitions $(p, e, \gamma, q, \gamma), \forall\gamma \in \Gamma$
  - Replace every call transition (p, i, q) in the RA with the PDA transitions $(p, \varepsilon, \gamma, q_i, q.\gamma), \forall\gamma \in \Gamma$
  - In each module of the RA, for each final state p add the PDA transition $(p, \varepsilon, q, q, \varepsilon)$ and de-finalize all final states except in module 0
  - For each final state p in module 0, add the additional PDA transition $(p, \varepsilon, \perp, p, \varepsilon)$

PDA → CFG

- Let $P = (Q, A, \Gamma, s, \delta, \bot, F)$ be a PDA that accepts by ES
- Define a CFG with non-terminals $\{S\} \cup (Q \times Q \times \Gamma)$ and for each $q \in Q$, the rule $S \to (s, q, \bot)$, add additional rules to prove this claim:

$\forall w \in A^*,\quad (p,\gamma) \overset{w}{\to} (q, \gamma_1\gamma_2 \cdots \gamma_k)$ iff
$(p,q,\gamma) \Rightarrow^* w.(q,q_1,\gamma_1)(q_1,q_2,\gamma_2)\cdots(q_{k-1},q_k,\gamma_k)$
for some $q_1, \cdots, q_k \in Q$

From this claim, it follows that $(s,\bot) \overset{w}{\to} (q,\varepsilon)$ iff $(s,q,\bot) \Rightarrow^* w$

- Additional rules:
  - $\forall (p,e,\gamma,q,\varepsilon) \in \delta$, add the rule: $(p,q,\gamma) \to e$
  - $\forall (p,e,\gamma,q,\gamma_1 \cdots \gamma_k) \in \delta$ (where $k \geq 1$) and $\forall q_1, \cdots, q_k \in Q$, add the rule:

$$(p,q_k,\gamma) \to e.(q,q_1,\gamma_1)(q_1,q_2,\gamma_2)\cdots(q_{k-1},q_k,\gamma_k)$$