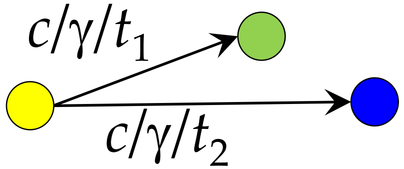
Visibly Pushdown Languages

- **Recall**: A PDA is a tuple $P = (Q, A, \Gamma, s, \delta, \bot, F)$ where
  - Finite set $\delta \subseteq Q \times (A \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^*$
  - ε-moves, multi-symbol push



Stack height is determined by the automaton's non-deterministic choices

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Key insight: To model the control flow of sequential computation in typical programming languages with nested, and potentially recursive, invocations of program modules, it seems natural to require the model to render its calls and returns visible.

Stack height is determined by the input word

Visibly Pushdown Automata

- A pushdown alphabet is a tuple $\hat{A} = \langle A_{\text{call}}, A_{\text{ret}}, A_{\text{int}} \rangle$
- A VPA is a tuple $V = \left(Q, \hat{A}, \Gamma, Q_0, \langle \delta_{\text{call}}, \delta_{\text{ret}}, \delta_{\text{int}} \rangle, \bot, F\right)$ where
  - Q is a finite set of states, Γ is a finite stack alphabet with $\bot \in \Gamma$
  - $Q_0 \subseteq Q$ is a set of initial states, $F \subseteq Q$ is a set of final states
  - $\delta_{\text{call}} \subseteq Q \times A_{\text{call}} \times Q \times (\Gamma - \{\bot\})$
  - $\delta_{\text{ret}} \subseteq Q \times A_{\text{ret}} \times \Gamma \times Q$
  - $\delta_{\text{int}} \subseteq Q \times A_{\text{int}} \times Q$

Single-letter push/pop,
no ε-moves, “unpoppable” $\bot$,
“cannot” read stack on A_{int}

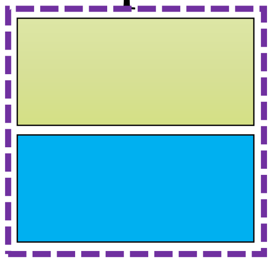
- Configuration: $(q, s) \in Q \times (\Gamma - \{\bot\})^*.\{\bot\}$; definition of “goes to”?
- $L(V) = \left\{ w \in A^* \middle| \exists q_0 \in Q_0, \exists q_f \in F \text{ such that } (q_0, \bot) \xrightarrow{w} (q_f, s) \right\}$

Accept by
FS only

Visibly Pushdown Languages (VPL)

possibly empty

- A language $L \subseteq A^*$ is a VPL if A can be partitioned into a pushdown alphabet $\hat{A}$ over which there is a VPA V such that $L = L(V)$
- *Examples*:
  - $\{a^n.b^n | n \geq 0\}$ is a VPL
  - $\{a^n.b.a^n | n \geq 0\}$ is a DCFL but not a VPL
- VPL has nicer closure properties than DCFL (and CFL)
- **Claim**: If L_1 and L_2 are VPL over the *same* pushdown alphabet $\hat{A}$, then $L_1 \cap L_2$ and $L_1 \cup L_2$ are VPL over $\hat{A}$
- **Proof**: Cross-product for intersection. For union?



Determinizability

- **Claim**: For every VPA V, there is a deterministic VPA V' over the same pushdown alphabet such that $L(V) = L(V')$
  - In a DVPA: $|Q_0| = 1$ and from each state, there is at most one transition on each $c \in A_{\text{call}}$, on each $a \in A_{\text{int}}$, and on each pair $(r, \gamma) \in A_{\text{ret}} \times \Gamma$
- Why does the subset construction fail?
  - **Attempt 1**: $Q' = 2^Q$, $\Gamma' = 2^\Gamma$
  - **Attempt 2**: $Q' = 2^Q$, $\Gamma' = 2^Q \times 2^\Gamma$ or perhaps $\Gamma' = 2^{Q \times \Gamma}$
- *Key observation*: We need to maintain a *summary* of what transitions are possible between a *call* and its corresponding *return*

VPA → DVPA (1/2)

- $Q' = 2^{Q \times Q} \times 2^{Q}$ and $Q_0' = \{(Id_Q, Q_0)\}$ where $Id_Q = \{(q, q) | q \in Q\}$
- $\Gamma' = \{(S, R, c) | (S, R) \in Q', c \in A_{\text{call}}\} \cup \{\perp\}$
- **Internal**: For every $a \in A_{\text{int}}$, $\big((S, R), a, (S', R')\big) \in \delta_{\text{int}}'$ where
$$S' = \{(p, q) | \exists q' \text{s.t.} (p, q') \in S \text{ and } (q', a, q) \in \delta_{\text{int}}\}$$
$$R' = \{q | \exists p \in R \text{ s.t.} (p, a, q) \in \delta_{\text{int}}\}$$
- *Key idea for calls/returns*: Postpone handling call-transitions that V does; just store the call actions and simulate the transitions corresponding to them at the time of the corresponding return
- **Call**: For every $c \in A_{\text{call}}$, $\Big((S, R), c, \big(Id_Q, R'\big), (S, R, c)\Big) \in \delta_{\text{call}}'$ where
$$R' = \{q | \exists p \in R, \exists \gamma \in \Gamma \text{ s.t.} (p, c, q, \gamma) \in \delta_{\text{call}}\}$$

VPA → DVPA (2/2)

- **Return 1**: For every $r \in A_{\text{ret}}$, $\big((S,R), r, (S',R',c), (S'',R'')\big) \in \delta'_{\text{ret}}$ where

$$U = \{(p,q) | \exists (q_1,q_2) \in S, \exists \gamma \in \Gamma \text{ s.t. } (p,c,q_1,\gamma) \in \delta_{\text{call}} \text{ and } (q_2,r,\gamma,q) \in \delta_{\text{ret}}\}$$
$$S'' = \{(p,q) | \exists q' \in S' \text{ s.t. } (p,q') \in S' \text{ and } (q',q) \in U\}$$
$$R'' = \{q | \exists p \in R' \text{ s.t. } (p,q) \in U\}$$

- **Return 2**: For every $r \in A_{\text{ret}}$, $\big((S,R), r, \bot, (S',R')\big) \in \delta'_{\text{ret}}$ where

$$S' = \{(p,q) | \exists q' \text{ s.t. } (p,q') \in S \text{ and } (q',r,\bot,q) \in \delta_{\text{ret}}\}$$
$$R' = \{q | \exists p \in R \text{ s.t. } (p,r,\bot,q) \in \delta_{\text{ret}}\}$$

- $F' = \{(S,R) | R \cap F \neq \emptyset\}$

- **Corollary**: VPL closed under complement

Decision Problems

- **Emptiness**: Given a VPA V, is $L(V) = \varnothing$?
  - VPA $\rightarrow$ pushdown system and compute $Post^*(\{(q_0, \bot) | q_0 \in Q_0\})$
- **Inclusion** (*undecidable for PDA*): Given VPA V_1 and V_2, is $L(V_1) \subseteq L(V_2)$?
  - $L(V_1) \subseteq L(V_2)$ iff $L(V_1) \cap \overline{L(V_2)} = \emptyset$
  - Determinize V_2, complement it, take cross-product with V_1 and check for emptiness
- **Equivalence** ($L(V_1) = L(V_2)$) and **Universality** ($L(V) = A^*$)?

A logical characterization of VPL

- Consider a word w and let $U = \{1, 2, \ldots, |w|\}$ denote the set of positions
- Define a binary relation μ over U such that $\mu(x, y)$ is true iff $w[x]$ is a call and $w[y]$ is its matching return
- Define the logic MSO_μ over $\hat{A}$ as:

$$\varphi \coloneqq Q_a(x) \mid x \in X \mid x \leq y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x.\varphi \mid \exists X.\varphi \mid \mu(x, y)$$

- Define $L(\varphi)$ as the set of words satisfying sentence φ

- **Theorem**: L is a VPL over $\hat{A}$ iff there is an MSO_μ sentence φ over $\hat{A}$ such that $L(\varphi) = L$

Unique minimized DVPA?

- Let $A = \{c_1, c_2, r, a, b\}$, $L_1 = L\big((b^*.a.b^*.a.b^*)^*\big)$, $L_2 = L\big((a^*.b.a^*.b.a^*)^*\big)$ and $L = \{c_1\}.L_1.\{r\} \cup \{c_2\}.L_2.\{r\}$
- Two non-isomorphic minimized DVPA for L: