Presburger formulas and Semilinear sets

A proof of their equivalence

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Overview

Notation

Corresponding operations on Presburger formulas and subsets of $\mathbb{N}^n$:

- $\vee$ - Conjunction and $\cup$ - Union (finite)
- $\wedge$ - Disjunction and $\cap$ - Intersection (finite)
- $\neg$ - Negation and $\mathbb{N}^n - X$ - Complementation
- $\exists$ - Universal quantifier and $\pi : \mathbb{N}^{n+1} \to \mathbb{N}^n$ - Projection

Partial Order on $\mathbb{N}^n$

We define a partial order on $\mathbb{N}^n$, denoted $\leq$.
Given $a, b \in \mathbb{N}^n$, we say $a \leq b$ if $a_i \leq b_i \; \forall \, 1 \leq i \leq n$.

Lemma: If $S \subseteq \mathbb{N}^n$ is such that elements in S are pairwise incomparable, then S is finite.

Proof: By induction on n.

Modified Presburger formulas

Definitions

The set of all (modified) Presburger formulas $\mathscr{P}$ is the smallest set satisfying

- $t_0 + \sum_{i=1}^{n} t_i x_i = t'_0 + \sum_{i=1}^{n} t'_i x_i$, where x_i for $1 \leq i \leq n$ are free variables, is in $\mathscr{P}$.
- If $\mathcal{P}_1$ and $\mathcal{P}_2$ are in $\mathscr{P}$, then $\mathcal{P}_1 \vee \mathcal{P}_2$ and $\mathcal{P}_1 \wedge \mathcal{P}_2$ are also in $\mathscr{P}$.
- If $\mathcal{P}$ is in $\mathscr{P}$, $\neg\mathcal{P}$ is in $\mathscr{P}$.
- If $\mathcal{P}(x_1, \ldots, x_n)$ is in $\mathscr{P}$, then $(\exists x_i)\mathcal{P}(x_1, \ldots, x_n)$ is in $\mathscr{P}$ for $1 \leq i \leq n$.

Presburger formulas

Examples

Notice that the following formulas are also in $\mathscr{P}$:

- $t_0 + \sum_{i=1}^{n} t_i x_i \leq t'_0 + \sum_{i=1}^{n} t'_i x_i$, equivalent to $(\exists z)(z + t_0 + \sum_{i=1}^{n} t_i x_i = t'_0 + \sum_{i=1}^{n} t'_i x_i)$
- $(\forall x_i)\mathcal{P}(x_1, \ldots, x_n)$, equivalent to $\neg(\exists x_i)(\neg\mathcal{P}(x_1, \ldots, x_n))$.

Some examples:

- $\mathcal{P}_1(x) = (\exists y)(x = y + 3 \ \wedge \ y < 5)$
- $\mathcal{P}_2(x, y, z) = (x + 3y = 4z + 7)$

Presburger sets

Definition and Examples

A set P is called a **Presburger set** in $\mathbb{N}^n$, $n > 0$ if $P = \{ (x_1, ..., x_n) \in \mathbb{N}^n \mid \mathcal{P}(x_1, ..., x_n) \}$, and we shall say the formula $\mathcal{P}$ describes the set P.

P is called a Presburger set if it is Presburger in some $\mathbb{N}^k, k > 0$.

Some examples:

- $P_1 = \{3, 4, 5, 6, 7\} = \{x \in \mathbb{N} \mid \mathcal{P}_1(x) \}$
- $P_2 = \{(7 + 4b - 3a, a, b) \mid a, b \in \mathbb{N}\} = \{x \in \mathbb{N}^3 \mid \mathcal{P}_2(x) \}$

Presburger sets

Closure Properties

Lemma: Presburger sets are closed under $\cup$, $\cap$, $\mathbb{N}^n - X$ and π.

Proof: Follows immediately from the definition of Presburger formulas, since the operations $\cup$, $\cap$, $\mathbb{N}^n - X$ and π correspond to $\vee, \wedge, \neg$ and $(\exists x_i)\mathcal{P}(x_1, ...x_n)$ on Presburger formulas, and Presburger formulas are closed under these operations.

Semilinear sets

Definitions

Let $C, P \subseteq \mathbb{N}^n$. Define $\mathcal{L}(C;P) \subseteq \mathbb{N}^n$ as

$$\mathcal{L}(C;P) := \{\, x_0 + \sum_{i=1}^{k} t_i x_i \mid x_0 \in C \text{ and } \forall 1 \leq i \leq k, t_i \in \mathbb{N}, x_i \in P\}.$$

C is called the set of constants, and P the set of periods.

$L \subseteq \mathbb{N}^n$ is **linear** if $L = \mathcal{L}(C;P)$ for a singleton set C and a finite set P.

$S \subseteq \mathbb{N}^n$ is **semilinear** if it is a finite union of linear sets, i.e.,

$S = \bigcup_{i=1}^{k} \mathcal{L}(C_i;P_i)$ for singleton sets C_i and finite sets P_i.

Semilinear sets

Examples

- The set $\mathbb{N}^n = \mathcal{L}(0^n; \{e_1, \ldots, e_n\})$ is semi-linear.
- The set $X \subseteq \mathbb{N}^2$ defined as $X = \{(x, y) | x \geq 1\}$ is semi-linear as $X = \mathcal{L}((1, 0); \{(1, 0), (0, 1)\})$
- The set $X \subseteq \mathbb{N}^n$ defined by $X = \{(a_1, \ldots, a_n) | a_1 \text{ not divisible by } 5\}$ is semi-linear because $X = \mathcal{L}(\{e_1, 2e_1, 3e_1, 4e_1\}; \{5e_1, e_2, \ldots, e_n\})$

Semilinear sets

Basic Properties

Lemma:

1. For $C_1, C_2, P \subseteq \mathbb{N}^n$,
   $\mathcal{L}(C_1 \cup C_2; P) = \mathcal{L}(C_1; P) \cup \mathcal{L}(C_2; P)$
2. For $C, P_1, P_2 \subseteq \mathbb{N}^n$,
   $\mathcal{L}(\mathcal{L}(C; P_1); P_2)) = \mathcal{L}(C; P_1 \cup P_2)$
3. For $C_1, P_1 \subseteq \mathbb{N}^k$ and $C_2, P_2 \subseteq \mathbb{N}^l$,
   $\mathcal{L}(C_1; P_1) \times \mathcal{L}(C_2; P_2) = \mathcal{L}(C_1 \times C_2; (P_1 \times \{0^l\}) \cup (\{0^k\} \times P_2))$

Proof: Follows from the definition.

Semilinear sets

Basic Properties

Corollaries:

- If $S \subseteq \mathbb{N}^n$ is semilinear and $P \subseteq \mathbb{N}^n$ is finite, then $\mathcal{L}(S; P)$ is semilinear.
  Follows by noticing that $S = \bigcup_{i=1}^{k} \mathcal{L}(C_i; P_i)$ and using lemmas 1 and 2 successively.
- If $X \subseteq \mathbb{N}^n$ and $Y \subseteq \mathbb{N}^m$ are semilinear, then $X \times Y \subseteq \mathbb{N}^{n+m}$ is semilinear.
  Follows from lemma 3 because the cartesian product distributes across union.

Semilinear sets

Closure under Linear Maps

Lemma: If $\tau : \mathbb{N}^n \to \mathbb{N}^m$ is a linear map, and $A \subseteq \mathbb{N}^n$ is semilinear, then $\tau(A)$ is semilinear.

Proof: As A is semilinear, $A = \bigcup_{i=1}^{k} \mathcal{L}(C_i; P_i)$. Then,

$$\tau(A) = \tau(\bigcup_{i=1}^{k} \mathcal{L}(C_i; P_i)) = \bigcup_{i=1}^{k} \tau(\mathcal{L}(C_i; P_i)) = \bigcup_{i=1}^{k} \mathcal{L}(\tau(C_i); \tau(P_i)))$$

Hence, $\tau(A)$ is semilinear.

Semilinear sets

Closure Properties

Theorem: Semilinear sets are closed under $\cup$, $\cap$, $\mathbb{N}^n - X$ and π.

Proof: Semilinear sets are clearly closed under $\cup$, since the finite union of semilinear sets is a finite union of a finite union of linear sets, which is simply a finite union of linear sets, and is hence semilinear.
The proofs for the remaining parts of the theorem are much longer.

Closure under Intersection

Let $L = \mathcal{L}(x_0; \{x_1, \ldots, x_p\})$, $L' = \mathcal{L}(x'_0; \{x'_1, \ldots, x'_q\})$ be linear sets.

Define $A = \{(y, z) \in \mathbb{N}^{p+q} \mid x_0 + \sum_{i=1}^{p} y_i x_i = x'_0 + \sum_{i=1}^{q} z_i x'_i\}$,

$$B = \{(y, z) \in \mathbb{N}^{p+q} \mid \sum_{i=1}^{p} y_i x_i = \sum_{i=1}^{q} z_i x'_i\}, \text{ and}$$

$\tau : \mathbb{N}^{p+q} \to \mathbb{N}^n$ as $\tau(y, z) = \sum_{i=1}^{p} y_i x_i$. τ is a linear map.

Closure under Intersection

Let C and P be the set of minimal elements in A and $B - \{0^n\}$. These sets are finite since their elements are pairwise incomparable. By arguments identical to those for basic Presburger sets, we have that $A = \mathcal{L}(C; P)$ and hence, A is semilinear.

$L \cap L' = \{x_0 + u \mid u \in \tau(A)\}$, and hence, $L \cap L'$ is semilinear.

Let $X = \bigcup_{i=1}^{n} L_i$ and $X' = \bigcup_{i=1}^{m} L'_i$ be two semilinear sets.

$X \cap X' = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} L_i \cap L_j$, a finite union of semilinear sets, and hence semilinear.

Closure under inverses of Linear Maps

Lemma: If $\tau : \mathbb{N}^n \to \mathbb{N}^m$ is a linear map, and $A \subseteq \mathbb{N}^m$ is semilinear, then $\tau^{-1}(A)$ is semilinear.

Proof: Define $\eta : \mathbb{N}^n \to \mathbb{N}^{n+m}$, $\eta(x) = (x, \tau(x))$. η is linear.
Hence, $\eta(\mathbb{N}^n)$ is semilinear. Since A is semilinear, $\mathbb{N}^n \times A$ is semilinear, and hence, $K = \eta(\mathbb{N}^n) \cap (\mathbb{N}^n \times A)$ is semilinear as well.
Define $\pi : \mathbb{N}^{n+m} \to \mathbb{N}^n$, $\pi(x, y) = x$. π is linear.
Hence $\pi(K)$ is semilinear. Note that $\pi(K) = \tau^{-1}(A)$, and hence, $\tau^{-1}(A)$ is semilinear.

Linear Independence of Periods

Lemma: Every linear set L can be written as the union of linear sets with linearly independent periods.

Proof: By induction on the number of elements in P. The statement is clearly true if L has only one period.
Assume the statement is true for linear sets with $\leq m$ periods.
Let L have $m+1$ periods, i.e., $L = \mathcal{L}(x_0; \{x_1, ..., , x_{m+1}\}$, and let the periods be linearly dependent. Then, by a relabelling of indices of the periods, $\exists k$, $1 \leq k \leq m$ such that for some $a_1, .., a_m \in \mathbb{N}$

$$\sum_{i=1}^{k} a_i x_i = \sum_{i>k}^{m} a_i x_i$$

Linear Independence of Periods

For each $j > k$, let $C_j = \{x_0 + b_j x_j \mid 0 \leq b_j < a_j\}$ if $a_j \geq 1$ and $\{x_0\}$ otherwise, and $P_j = P - \{x_j\}$. Define $Z_j = \mathcal{L}(C_j, P_j)$

Claim : $L = \bigcup_{j>k} Z_j \ (= Z)$

Clearly, by definition, $Z_j \subseteq L$ for each $j > k$ and hence, $Z \subseteq L$.

Let $y = x_0 + \sum_{i=1}^{m} b_i x_i$. If $b_j \geq a_j$ for all $j > k$, then

$$y = x_0 + \sum_{i=1}^{m} b_i x_i + \sum_{i=1}^{k} a_i x_i - \sum_{i>k}^{m} a_i x_i$$

$$= x_0 + \sum_{i=1}^{k} (b_i + a_i) x_i + \sum_{i>k}^{m} (b_i - a_i) x_i$$

Linear Independence of Periods

Thus, we can assume that $b_j < a_j$ for some $j > k$. Then, $y = x_0 + b_j x_j + \sum_{i \neq j} b_i x_i \in Z_j$ and hence, $L \subseteq Z$, giving $L = Z$. Now each Z_j has $\leq m$ periods and by the induction hypothesis, can be written as a union of sets with linearly independent periods. Hence, L can be written as a union of sets with linearly independent periods and the lemma follows.

Span of a Basis

Definition: Given $\{x_1, .., x_k\} \subseteq \mathbb{N}^n$,

$$span_{\mathbb{Z}}\{x_1, .., x_k\} = \{\sum_{i=1}^{k} a_i x_i \mid a_i \in \mathbb{Z}\, \forall\, 1 \leq i \leq k\}$$

Lemma: Let $\{x_1, .., x_n\}$ be a set of linearly independent vectors in $\mathbb{N}^n$. Then, $\exists\, k_0 \in \mathbb{N}^n$ such that $\forall y \in \mathbb{N}^n$, $\exists k$ such that $ky \in span_{\mathbb{Z}}\{x_1, .., x_n\}$ and $1 \leq k \leq k_0$.

Proof: It is enough to show that for each $e_i = (0, \ldots 0, 1, 0 \ldots 0)$ (1 at the ith coordinate), $\exists\, k_i$ such that $k_i e_i \in span_{\mathbb{Z}}\{x_1, ..., x_n\}$. We show this for $i = 1$, as the same proof works for all i's. Let $\pi : \mathbb{N}^n \to \mathbb{N}^{n-1}$ be the projection onto the last $n-1$ coordinates. π is linear.

Span of a Basis

$\{\pi(x_1), ..., \pi(x_n)\} \subseteq \mathbb{N}^{n-1} \subseteq \mathbb{Q}^{n-1}$. This is a set of n vectors in a vector space of dimension $n-1$, and is hence linearly dependent.

Hence $\sum_{i=1}^{n} q_i \pi(x_i) = 0$ for some $\{q_i\}$, with atleast one non-zero q_i.

Clearing denominators, we get $\sum_{i=1}^{n} r_i \pi(x_i) = 0$, where $\{r_i\} \subseteq \mathbb{Z}^{n-1}$.

$\sum_{i=1}^{n} r_i \pi(x_i) = \pi(\sum_{i=1}^{n} r_i x_i) = 0 \implies \sum_{i=1}^{n} r_i x_i = (k_1, 0, ..., 0)$.

As the x_i's are linearly independent, $k_1 \neq 0$.

Thus, $k_1 e_1 \in span_{\mathbb{Z}}\{x_1, \ldots, x_n\}$.

Span of a Basis

Lemma: Let $\{x_1, .., x_n\} \subseteq \mathbb{N}^n$ be a set of linearly independent vectors and $y \in \mathbb{N}^n$. Let k_y denote the smallest k such that $ky \in span_{\mathbb{Z}}\{x_1, ..., x_n\}$. Let $k_y y = \sum_{i=1}^{n} a_i x_i$ and $ky = \sum_{i=1}^{n} b_i x_i$.
Then, $\exists\, p \in \mathbb{N}$ such that $k = pk_y$ and $b_i = pa_i$.

Proof: Let $k = pk_y + r$, with $r < k_y$. Then, $ry = \sum_{i=1}^{n} (b_i - pa_i) x_i$.
Hence, $ry \in span_{\mathbb{Z}}\{x_1, ..., x_n\}$, giving $r = 0$.
The linear independence of $\{x_i\}$ gives $b_i - pa_i = 0 \;\forall\, 1 \leq i \leq n$.

Closure under Complements

Lemma: let $L = \mathcal{L}(0^n; \{x_1, ..., x_{j_0}\})$ be a linear subset of $\mathbb{N}^n$ with independent periods. Then $\mathbb{N}^n - L$ is semilinear.

Proof: Let $P = \{x_1, ..., x_{j_0}\}$ be the set of independent periods of the X. Adjoin $n - j_0$ of the e_i's to P to get n linearly independent vectors in $\mathbb{N}^n$. Let k_0 be as in the previous lemma.

Closure under Complements

Define the following sets:

- $G_1 = \{y \in \mathbb{N}^n \mid k_y y = \sum_{i=1}^{n} a_i x_i \Rightarrow \exists\, i$ such that $a_i < 0\}$ and $H_1 = \mathbb{N}^n - G_1$
- $G_2 = \{y \in \mathbb{N}^n \mid k_y y = \sum_{i=1}^{n} a_i x_i \Rightarrow \exists\, i > j$ such that $a_i > 0\}$ and $H_2 = H_1 - G_2$

Note that both $G_1, G_2 \subseteq \mathbb{N}^n - L$. Hence,

$$\mathbb{N}^n - L = (G_1 \cup G_2) \cup ((\mathbb{N}^n - (G_1 \cup G_2)) - L)$$
$$= G_1 \cup G_2 \cup ((H_1 - G_2) - L) = G_1 \cup G_2 \cup (H_2 - L).$$

Hence, to show that $\mathbb{N}^n - L$ is semilinear, it suffices to show that G_1, G_2 and $H_2 - L$ are semilinear.

Closure under Complements

Define the following linear maps:

- For $1 \leq k \leq k_0$ and $I \subseteq \{1, \ldots, n\}$, define $\tau_{k,I} : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{N}^n \times \mathbb{N}^n$, $\tau_{k,I}(y, a) = (ky + \sum_{i \in I} a_i x_i, \sum_{i \notin I} a_i x_i)$
- $\pi : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{N}^n$, $\pi(x, y) = x$

Define the following subsets of $\mathbb{N}^n \times \mathbb{N}^n$:

- $K = \{(y, y) \mid y \in \mathbb{N}^n\}$. Let $T : \mathbb{N}^n \to \mathbb{N}^n \times \mathbb{N}^n$, $T(y) = (y, y)$. Then T is linear, and $K = T(\mathbb{N}^n)$ and is hence semilinear.
- $D_i = \{(y, a) \mid a_i > 0\}$. $D_i = \mathcal{L}(e_{n+i}; \{e_1, \ldots, e_{2n}\})$ and hence, D_i is semilinear.
- $A_I = \{(y, a) \mid a_i > 0 \; \forall \, i \in I\}$. $A_I = \mathcal{L}(\sum_{i \in I} e_{n+i}; \{e_1, \ldots, e_{2n}\})$ and hence, A_I is semilinear.

Closure under Complements

First, we show G_1 is semilinear.
$\tau_{k,I}^{-1}(K)$ and A_I are semilinear, thus $\tau_{k,I}^{-1}(K) \cap A_I$ is semi-linear, and $\pi(\tau_{k,I}^{-1}(K) \cap A_I)$ is semilinear as well.
$G_1 = \bigcup_{k \leq k_0} \bigcup_{I \neq \phi} \pi(\tau_{k,I}^{-1}(K) \cap A_I)$ and hence, G_1 is semilinear.
Next, we show G_2 is semilinear.
D_i is semilinear, thus $D_i \cap \tau_{k,I}^{-1}(K) \cap A_I$ is semi-linear, and $\pi(D_i \cap \tau_{k,I}^{-1}(K) \cap A_I)$ is semilinear as well.
$G_2 = \bigcup_{i > j_0} \bigcup_{I \subseteq \{1,\ldots,n\} - \{i\}} \bigcup_{k \leq k_0} \pi(D_i \cap A_I \cap \tau_{k,I}^{-1}(K))$ and hence, G_2 is semilinear.

Closure under Complements

It remains to prove $H_2 - L$ is semilinear.
For $k \leq k_0$ and $j \leq j_0$, define

$$B_{kj} = \{(y, b) \in \mathbb{N}^n \times \mathbb{N}^{j_0} \mid k_y y = \sum_{i=1}^{j} a_i x_i, a_j \text{ not divisible by } k_y\}.$$

Define $E_k = \{(y, a) \in \mathbb{N}^n \times \mathbb{N}^{j_0} \mid k_y y = \sum_{i=1}^{j} b_i x_i\}$ and
$F_{kj} = \{(y, b) \mid b_j \text{ is not divisible by } k\}$. E_k and F_{kj} are semilinear.
Also, $B_{kj} = E_k \cap F_{kj}$ and thus B_{kj} is semilinear. Now,
$H_2 - X = \bigcup_{j \leq j_0} \bigcup_{k \leq k_0} \pi(B_{kj})$. Hence, $H_2 - L$ is semi-linear and thus
we are done.

Closure under Complements

Lemma: let $L \subseteq \mathbb{N}^n$ be a linear subset with independent periods. Then $\mathbb{N}^n - L$ is semi-linear.

Proof: Let $L = \mathcal{L}(x_0; \{x_1, \ldots, x_k\})$ with $k \leq n$.
For each i such that $(x_0)_i > 0$, define $C_i = \{(u_1, \ldots, u_n) \mid u_i = 0$ for $j \neq i$ and $0 \leq u_j < (x_0)_i\}$ and $P_i = \{e_1, ..., e_{i-1}, e_{i+1}, ..., e_n\}$.
Then, $Z_i = L(C_i; P_i)$ is semilinear and $G = \bigcup_{(x_0)_i > 0} Z_i$ is the set of all elements in $\mathbb{N}^n$ such that $y \geq x_0$ is false. Thus, $G \subseteq \mathbb{N}^n - L$.
Let $Y = \{y \in \mathbb{N}^n \mid x_0 \leq y\}$. $Y = \mathbb{N}^n - G$, and hence,
$\mathbb{N}^n - L = G \cup (\mathbb{N}^n - L - G) = G \cup (Y - L)$.

Closure under Complements

Now, we show $Y - L$ is semilinear.
Define $f : \mathbb{N}^n \to Y$, $f(y) = y + x_0$. Note that $f(\mathcal{L}(C; P)) = \mathcal{L}(f(C); P)$ and hence, Z is semilinear $\Rightarrow f(Z)$ is semilinear. We have that
$f^{-1}(Y - L) = f^{-1}(Y) - f^{-1}(L) = \mathbb{N}^n - f^{-1}(L)$.
$f^{-1}(L) = L(0^n; \{x_1, \ldots, x_n\})$ and thus $\mathbb{N}^n - f^{-1}(L)$ is semilinear by the previous lemma. Since $Y - L \subseteq f(\mathbb{N}^n)$,
$Y - L = f(f^{-1}(Y - L))$, and hence, $Y - L$ is semilinear.

Closure under Complements

Theorem: Let $Y \subseteq \mathbb{N}^n$ be a semilinear subset. Then $\mathbb{N}^n - Y$ is a semilinear subset of $\mathbb{N}^n$.

Proof: It is enough to consider the case when Y is linear. By the lemma on independent periods, $Y = \bigcup_{i=1}^{m} Z_i$ with each Z_i being linear with independent periods. Then, $\mathbb{N}^n - Y = \bigcap_{i=1}^{m} (\mathbb{N}^n - Z_i)$. By the previous lemma, each $\mathbb{N}^n - Z_i$ is semilinear, and the intersection of semilinear sets is semilinear. Hence, $\mathbb{N}^n - Y$ is semilinear.

Semilinear sets are Presburger sets

We will now show that every Semilinear set is Presburger.
Let $L \subseteq \mathbb{N}^n$ be linear. Then, $L = \mathcal{L}(v_0; \{v_1, ..., v_k\})$.
Let v_{ij}, $0 \leq i \leq k, 1 \leq j \leq n$ denote the j^{th} coordinate of v_i.
Define $\mathcal{P}_L(x_1, ..., x_n) := (\exists a_1)...(\exists a_k) \left(\bigwedge_{j=1}^{n} (x_j = v_{0j} + \sum_{i=1}^{k} a_i v_{ij}) \right)$
$\mathcal{P}_L$ describes L, and hence L is a Presburger set.
If S is semilinear, then $S = \bigcup_{i=1}^{k} L_i$ for linear sets L_i.
Define $\mathcal{P}_S := \bigvee_{i=1}^{k} \mathcal{P}_{L_i}$, and $\mathcal{P}_S$ describes S.

Presburger Sets are Semilinear Sets

Outline of proof

We will now show that every Presburger set is Semilinear.
We call a Presburger set $B \subseteq \mathbb{N}^n$ a *basic* Presburger set if B is described by a Presburger formula $\mathcal{P}$, where

$$\mathcal{P} = \left(t_0 + \sum_{i=1}^{n} t_i x_i = t_0' + \sum_{i=1}^{n} t_i' x_i \right).$$

We show that basic Presburger sets are semilinear. All Presburger sets are obtained through operations $\cup$, $\cap$, $\mathbb{N}^n - X$ and π on basic Presburger sets, and since semilinear sets are closed under $\cup$, $\cap$, $\mathbb{N}^n - X$ and π, all Presburger sets are semilinear.

Basic Presburger sets are Semilinear

Let B be a basic Presburger set described by
$$\mathcal{P} = \left(t_0 + \sum_{i=1}^{n} t_i x_i = t'_0 + \sum_{i=1}^{n} t'_i x_i \right).$$
Let $\mathcal{P}'$ be the corresponding homogeneous formula,
$$\mathcal{P}' = \left(\sum_{i=1}^{n} t_i x_i = \sum_{i=1}^{n} t'_i x_i \right)$$
Let C be the set of minimal solutions to $\mathcal{P}$ and P be the set of minimal solutions to $\mathcal{P}'$ in $\mathbb{N}^n - \{0^n\}$. C and P are finite, since their elements are pairwise incomparable.

Basic Presburger sets are Semilinear

Claim: $B = \mathcal{L}(C; P)$ and hence, B is semilinear.
Proof: Solutions to $\mathcal{P}'$ are closed under addition, and if v solves $\mathcal{P}$ and u solves $\mathcal{P}'$, $v + u$ solves $\mathcal{P}$. Hence, $\mathcal{L}(C; P) \subseteq B$.
Now, assume v solves $\mathcal{P}$. Then $\exists\, v' \in C$ such that $v' \leq v$.
Further, $v - v'$ solves $\mathcal{P}'$. We show that any solution of $\mathcal{P}'$ is in $\mathcal{L}(0^n; P)$, and hence $B \subseteq \mathcal{L}(C; P)$.

Basic Presburger sets are Semilinear

Let $v \in \mathbb{N}^n$ solve $\mathcal{P}'$. We show that $v \in \mathcal{L}(0^n; P)$.

The proof is by induction on $s = \sum_{i=1}^{n} v_i$.

For $s = 0$, $0^n \in \mathcal{L}(0^n; P)$.
Assume that for $s \leq k$, v solves $\mathcal{P}' \Rightarrow v \in \mathcal{L}(0^n; P)$.
Let v be such that $s = k + 1$. Then, $\exists\, v' \in P$ such that $v' \leq v$, $v' \neq 0^n$. Then $v - v'$ has $s \leq k$ and solves $\mathcal{P}'$, and hence, $v - v' \in \mathcal{L}(0^n; P)$. Hence, $v \in \mathcal{L}(0^n; P)$ as well.

This completes the proof of equivalence of Semilinear sets and Presburger sets.

References

1. Ginsburg, Seymour; Spanier, Edwin H. "Bounded Algol-Like Languages." Transactions of the American Mathematical Society, vol. 113, no. 2, 1964, pp. 333–368.
2. Ginsburg, Seymour; Spanier, Edwin H. "Semigroups, Presburger formulas, and languages". Pacific J. Math. 16 (1966), no. 2, 285-296.