Reachability in Pushdown Systems

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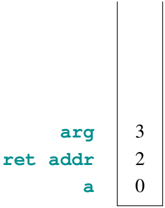
Outline

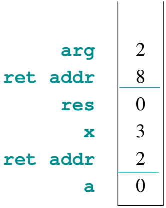
Modelling programs with procedures

```
0 void main() {
1     int a;
2     a = fact(3);
  }

3 int fact(int x) {
4     int res;
5     if (x==1)
6       res = 1;
7     else
8       res = x * fact(x-1);
9     return res;
  }
```



After call to fact in line 2

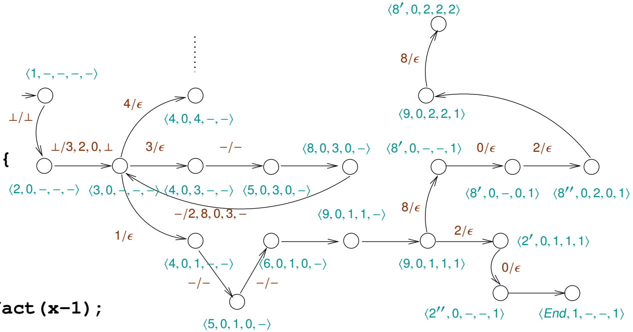


After call to fact in line 8

Pushdown system induced by program

```
0  void main() {
1     int a;
2     a = fact(3);
   }

3  int fact(int x) {
4     int res;
5     if (x==1)
6       res = 1;
7     else
8       res = x * fact(x-1);
9     return res;
   }
```



State representation: ⟨ stmt no, a, x, res, rval ⟩

Pushdown Systems

A pushdown system is of the form

$$\mathcal{P} = (P, \Gamma, \Delta)$$

where

- P is a finite set of states
- Γ is the stack alphabet,
- $\Delta \subseteq P \times \Gamma \times P \times \Gamma^*$ is the non-deterministic transition relation.
  - Each transition is of the form $pa \to q\gamma$.

A pushdown system is thus like a PDA but with no input and no initial/final states.
Can model several useful classes of systems

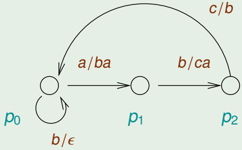
- PDA with input abstracted away
- Programs with finite state but with procedure calls (or "Boolean Programs")

Example Pushdown System

Example pushdown system $\mathcal{P}_1$

$$
\begin{array}{lcl}
p_0 a & \rightarrow & p_1 ba \\
p_1 b & \rightarrow & p_2 ca \\
p_2 c & \rightarrow & p_0 b \\
p_0 b & \rightarrow & p_0 \epsilon.
\end{array}
$$

Diagram representation



Sequence of configurations reachable from $p_2 cbba$:

$$p_2 cbba \overset{1}{\Rightarrow} p_0 bbba \overset{1}{\Rightarrow} p_0 bba \overset{1}{\Rightarrow} p_0 ba \overset{1}{\Rightarrow} p_0 a \overset{1}{\Rightarrow} p_1 ba \overset{1}{\Rightarrow} p_2 caa \overset{1}{\Rightarrow} p_0 baa \overset{1}{\Rightarrow} p_0 aa$$

Configuration graph induced by a pushdown system

$\mathcal{P}$ induces a (possibly infinite) graph whose

- nodes are configurations of $\mathcal{P}$ represented by strings in $P \cdot \Gamma^*$
- edges are $c \rightarrow c'$ iff $c \overset{1}{\Rightarrow} c'$ in $\mathcal{P}$.

Given a set of configurations C of $\mathcal{P}$ we can define

$$Pre^*(C) = \{c \mid \exists c' \in C : c \overset{*}{\Rightarrow} c'\}.$$

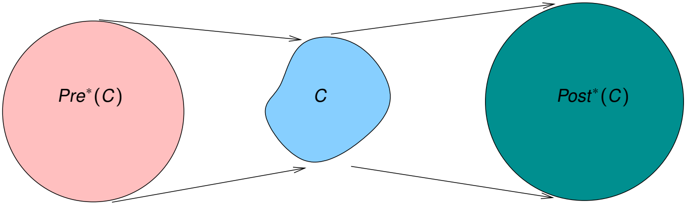
And similarly

$$Post^*(C) = \{c' \mid \exists c \in C : c \overset{*}{\Rightarrow} c'\}.$$

Reachability in Pushdown Systems

Theorem (Büchi, 1964)

Let $\mathcal{P}$ be a pushdown system, and let C be a regular set of configurations of $\mathcal{P}$. Then $Pre^*(C)$ and $Post^*(C)$ are also regular sets. Moreover given an NFA for C we can construct an NFA accepting $Pre^*(C)$ and $Post^*(C)$ respectively.

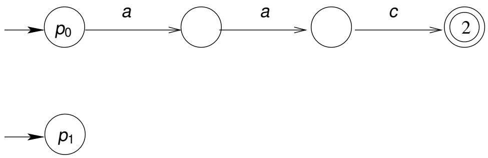


Saturation algorithm for Pre^*

Let $\mathcal{P} = (P, \Gamma, \Delta)$ be a pushdown system, and C be a set of configurations of $\mathcal{P}$.
A P-automaton for C is an NFA $\mathcal{A} = (Q, \Gamma, P, \Delta', F)$ that accepts from an initial state $p \in P$ exactly the words w such that $pw \in C$.

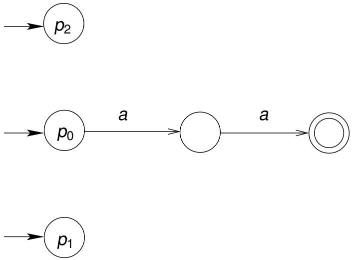
- The control states of $\mathcal{P}$ are used as initial states of $\mathcal{A}$.
- $\mathcal{A}$ must not have a transition to an initial state.

Example P-automaton for $\{p_0 aac\}$:



Example P-automaton

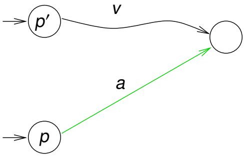
Example P-automaton for $\{p_0 aa\}$:



Saturation algo for Pre^*

Input: Pushdown system $\mathcal{P} = (P, \Gamma, \Delta)$, and P-automaton $\mathcal{A}$ for C.
Output: $\overline{\mathcal{A}}$ accepting $Pre^*(C)$.

- Repeat until no more new edges can be added to $\mathcal{A}$:
  - If $pa \to p'v \in \Delta$ and $p' \xrightarrow{v} q$ in $\mathcal{A}$, then add $p \xrightarrow{a} q$ to $\mathcal{A}$.



- Return $\overline{\mathcal{A}}$.

Run saturation algo for Pre^*

Example pushdown system

$$
\begin{array}{rcl}
p_0 a & \rightarrow & p_1 ba \\
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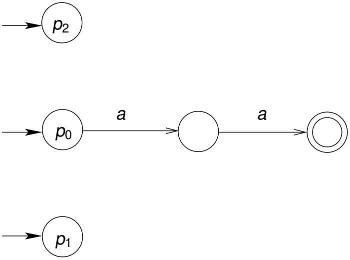
\end{array}
$$

P-automaton for $C = \{p_0 aa\}$:



Run saturation algo for *Pre**

Example pushdown system

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\begin{array}{rcl}
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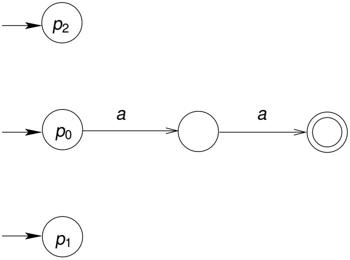
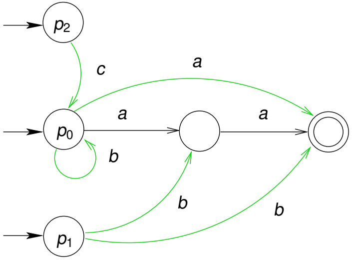
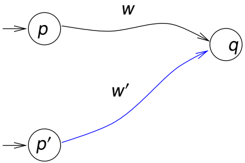


Saturated P-automaton:



Correctness

- $Pre^*(C) \subseteq L(\overline{\mathcal{A}})$.
  - Prove by induction on n that $pw \overset{n}{\Rightarrow} p'w' \in C$ implies $pw \in L(\overline{\mathcal{A}})$.
- $L(\overline{\mathcal{A}}) \subseteq Pre^*(C)$.
  - Let $\mathcal{A}_i$ be P-automaton after i-th step of algo.
  - Claim 1: If $pw \in L(\mathcal{A}_i)$ then $pw \overset{*}{\Rightarrow} p'w' \in C$.
    - Proof by induction on i runs into rough weather; Need to argue about path segments, not just accepting paths.
  - Strengthen Claim to: If $p \xrightarrow{w} q$ in $\mathcal{A}_i$ then there exists $p'w'$ such that $p' \xrightarrow{w'} q$ in $\mathcal{A}$ and $pw \overset{*}{\Rightarrow} p'w'$.

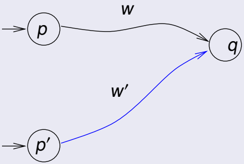


  - Observe that strengthened Claim implies Claim 1 and completes proof.

Proof of Claim

Claim

If $p \xrightarrow{w} q$ in $\mathcal{A}_i$ then there exists $p'w'$ such that $p' \xrightarrow{w'} q$ in $\mathcal{A}$ and $pw \overset{*}{\Rightarrow} p'w'$.

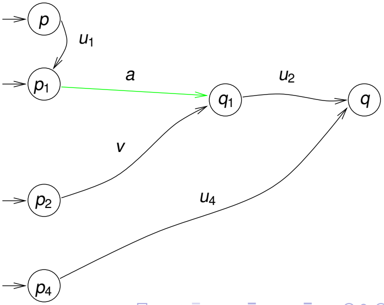


Proof: By induction on i. For the induction step, suppose we added the edge (p_1, a, q_1) in $\mathcal{A}_{i+1}$ due to the PDA transition $p_1 a \rightarrow p_2 v$. Suppose $p \xrightarrow{w} q$ in $\mathcal{A}_{i+1}$.

Proof of Claim - II

If this path does not use the new edge, it is a path in $\mathcal{A}_i$ itself and by induction hypothesis we are done.
If it uses the new edge 1 or more times consider the representative case of when it uses it exactly once. Say the path is
$p \xrightarrow{u_1} p_1 \xrightarrow{a} q_1 \xrightarrow{u_2} q$.

- By IH there has to be a path $p_3 u_3$ to p_1 in $\mathcal{A}$ such that $pu_1 \overset{*}{\Rightarrow} p_3 u_3$. But since $\mathcal{A}$ has no incoming edges to the P-states, we must have $p_3 = p_1$ and $u_3 = \epsilon$. So $pu_1 \overset{*}{\Rightarrow} p_1$.
- By IH we also have a path $p_4 u_4$ to q in $\mathcal{A}$ such that $p_2 v u_2 \overset{*}{\Rightarrow} p_4 u_4$.
- Putting these together: $pw = pu_1 a u_2 \overset{*}{\Rightarrow} p_1 a u_2 \overset{1}{\Rightarrow} p_2 v u_2 \overset{*}{\Rightarrow} p_4 u_4$, and $p_4 u_4$ is the required $p'w'$.



Proof of Claim - III (General argument)

By a second induction on the number of times the path $p \xrightarrow{w} q$ uses the edge $p_1 \xrightarrow{a} q_1$, we prove that $\exists p'w'$ such that $pw \stackrel{*}{\Rightarrow} p'w'$ and $p' \xrightarrow{w'} q$ in $\mathcal{A}$. Base: If w does not use the new edge, it is a path in $\mathcal{A}_i$ itself and by IH-1 we are done. Ind-step: If it uses the new edge $k+1$ times, let the path be $p \xrightarrow{u_1} p_1 \xrightarrow{a} q_1 \xrightarrow{u_2} q$. where u_2 does not use the new edge.

- Since $p \xrightarrow{u_1} p_1$ uses the new edge k times, by IH-2 there is a path $p_3 u_3$ to p_1 in $\mathcal{A}$ such that $pu_1 \stackrel{*}{\Rightarrow} p_3 u_3$. But since $\mathcal{A}$ has no incoming edge to P-states, we have $p_3 = p_1$ and $u_3 = \epsilon$. So $pu_1 \stackrel{*}{\Rightarrow} p_1$.
- By IH-1 we also have a path $p_4 u_4$ to q in $\mathcal{A}$ such that $p_2 v u_2 \stackrel{*}{\Rightarrow} p_4 w_4$.
- Thus: $pw = pu_1 a u_2 \stackrel{*}{\Rightarrow} p_1 a u_2 \stackrel{1}{\Rightarrow} p_2 v u_2 \stackrel{*}{\Rightarrow} p_4 u_4$, and $p_4 u_4$ is the required $p'w'$.