Class of regularity preserving functions

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u.p. reducible functions

A function $f: \mathbb{N} \longrightarrow \mathbb{N}$, is u.p. reducible if,

for every modulus m, there is a period p s.t. for all but finitely many $n \in \mathbb{N}$

$$f(n) \equiv f(n+p)(mod\ m)$$

Essentially increasing function

A function $f: \mathbb{N} \longrightarrow \mathbb{N}$, is essentially increasing if,

for every k, $f(n) \geq k$ for all but finitely many $n \in \mathbb{N}$

Notations

- $F = \{f \mid f\ is\ essentially\ increasing\ and\ u.p.reducible\}$
- $G = \{f \mid f\ is\ u.p.preserving\}$
- $H = \{f \mid f\ is\ u.p.reducible\}$

$F \subset G$

- Let f be essentially increasing and u.p. reducible and A is u.p.
- Take n_0, m such that $(\forall n \geq n_0)(n \in A \Leftrightarrow n + m \in A)$
- Then, $(\forall n_1, n_2 \geq n_0)\big(n_1 \equiv n_2\ (mod\ m) \Rightarrow (n_1 \in A \Leftrightarrow n_2 \in A)\big)$
- Take n_0', p s.t. $(\forall n \geq n_0')\ (f(n) \geq n_0\ \&\ f(n) \equiv f(n+p)(mod\ m))$
- Then,

$$\begin{aligned} n \geq n_0' &\Rightarrow f(n), f(n+p) \geq n_0\ \& f(n) \equiv f(n+p)(mod\ m) \\ &\Rightarrow f(n) \in A \Leftrightarrow f(n+p) \in A \\ &\Rightarrow n \in f^{-1}(A) \Leftrightarrow n + p \in f^{-1}(A) \end{aligned}$$

- So, $f^{-1}(A)$ is u.p.

$F \neq G$

- Define

$$f(n) = \begin{cases} 0 & if\ n\ is\ even \\ n! & o.w. \end{cases}$$

- f is not essentially increasing and hence $f \notin F$.
- And if A is u.p., $f^{-1}(A)$ contains either all or no even numbers; and either only finitely many or all but finitely many odd numbers
- So, $f^{-1}(A)$ is u.p. and $f \in G$

$G \subset H$

- Let f be u.p. preserving and $m \in \mathbb{N}$
- $f^{-1}(\{j | j \equiv n (mod\ m)\})$ is u.p. for $0 \leq n \leq m - 1$
- If p is common period for these m u.p. sets, then

$$f(n) \equiv f(n+p)(mod\ m)$$

for all but finitely many n.

$G \neq H$

- For any set A, s.t. A is non u.p., define
$$f(n) = \begin{cases} 0 & if\ n \in A \\ n! & o.w. \end{cases}$$
- For every m and every $n \geq m$,
$$f(n) \equiv f(n+1) \equiv 0 (mod\ m)$$
- So $f \in H$. But {0} is u.p. and $f^{-1}(\{0\}) = A$ is not u.p., so $f \notin G$

$H \supseteq \{f|P(f,L) is\ regular\ for\ every\ regular\ L \subseteq \{1\}^*\}$

- Let f: $\mathbb{N} \longrightarrow \mathbb{N}$ be any function. Then,

$$g = n + f(n)$$

is essentially increasing function.

- Let $P(f,L) is\ regular\ for\ every\ regular\ L \subseteq \{1\}^*$ and let A is u.p.
- Then,

$$g^{-1}(A) = \{i|g(i) \in A\}$$
$$\Longrightarrow g^{-1}(A) = \{i|i + f(i) \in A\}$$
$$\Longrightarrow g^{-1}(A) = \{|x||x \in P(f,\{z \in \{1\}^*||z| \in A\})\}$$

is u.p. so $g \in G \subseteq H, so\ f \in H$ (H is closed under sum and difference (if resulting function is $\mathbb{N} \longrightarrow \mathbb{N}$).

$$H \subseteq \{f|P(f,L) is\ regular\ for\ every\ regular\ L \subseteq \{1\}^*\}$$

- Let f be u.p. reducible. Then, $g \in F$, and so, g is u.p. preserving
- Let $L \subseteq \{1\}^*$ is regular. Now

$$\begin{aligned} P(f,L) &= \{x|\exists y\ (f(|x|) = |y| \& \ xy \in L)\} \\ &= \{x|\exists y\ (g(|x|) = |xy| \& \ xy \in L)\} \\ &= \{x \in \{1\}^* ||x| \in g^{-1}(\{|z||z \in L\})\} \end{aligned}$$

is regular

$G = \{f \in H | f^{-1}(\{j\}) is\ u.p. for\ every\ j\}$

- Let $f \in G$, then $f \in H$, and since {j} is u.p. for each j, $f^{-1}(\{j\})$ is u.p. for each j.
- For other side, let $f \in H$ and $f^{-1}(\{j\})$ is u.p. for every j. and let A is u.p.
- Let n_0, p be s.t.

$$(n \geq n_0)(n \in A \Leftrightarrow n + p \in A)$$

- Then,

$$\{j \in A | j \geq n_0\} = \left(\bigcup_{\substack{n \in A \\ n_0 \leq n < n+p}} \{j | j \equiv n (mod\ p)\} \right) - \{j | j < n_0\}$$

- Then,

$$f^{-1}(A) = f^{-1}(\{j \in A | j < n_0\}) \cup f^{-1}(\{j \in A | j \geq n_0\})$$

$$= f^{-1}(\{j \in A | j < n_0\}) \cup f^{-1}\left(\bigcup_{\substack{n \in A \\ n_0 \leq n < n+p}} \{j | j \equiv n (mod\ p)\}\right) - \{j | j < n_0\}$$

$$= \left(\bigcup_{\substack{j \in A \\ j < n_0}} f^{-1}(\{j\})\right) \cup \left(\left(\bigcup_{\substack{n \in A \\ n_0 \leq n < n+p}} \{i | f(i) \equiv n (mod\ p)\}\right) - \left(\bigcup_{j < n_0} f^{-1}(\{j\})\right)\right)$$