Regularity Preserving Relations

Ankit Kurani, Saurabh Awinashe

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1. Motivation
2. Introduction
3. Characterization as u.p. preserving relations
4. U.p. degenerating relations
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Prefix removals of regular languages

- Languages that consist of prefixes of strings from another language can be constructed.

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- Examples :
  1. FIRST-HALF(L) $= \{x \mid \exists y(|y| = |x| \ \& \ xy \in L)\}$
  2. SQUARE-ROOT(L) $= \{x \mid \exists y(|xy| = |x|^2 \ \& \ xy \in L)\}$

Prefix removals of regular languages

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- We need a characterization which can tell us whether a given prefix removal of a regular language is regular.

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Definitions

Definition 1

For each binary relation r on N and each language L,
$P(r, L) = \{x \mid \exists y(r(|x|, |y|) \;\&\; xy \in L)\}$

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Recall

A set $A \subseteq N$ is ultimately periodic(u.p.) if there exists $p \geq 1$ and $n_o \geq 0$ such that
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$r^{-1}(A) = \{i \mid (\exists j \in A)\ r(i, j)\}$ is u.p. for every u.p. set A.

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3. $r^{-1}(A) = A$

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Characterization as u.p. preserving relations

Lemma 1

If L is regular, then $\{|x| \mid x \in L\}$ is u.p.

Characterization as u.p. preserving relations

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If L is regular, then $\{|x| \mid x \in L\}$ is u.p.

Lemma 2

If A is u.p., then $\{x \in \Sigma^* \mid |x| \in A\}$ is regular for every finite Σ

Characterization as u.p. preserving relations

Theorem 1

A relation is regularity preserving iff it is u.p. preserving

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Proof ($\rightarrow$)

Suppose r is regularity preserving. Let A be any u.p.

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A relation is regularity preserving iff it is u.p. preserving

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Suppose r is regularity preserving. Let A be any u.p.

1. $L = \{1^n \mid n \in A\}$ is regular.

Characterization as u.p. preserving relations

Theorem 1

A relation is regularity preserving iff it is u.p. preserving

Proof ($\to$)

Suppose r is regularity preserving. Let A be any u.p.

1. $L = \{1^n \mid n \in A\}$ is regular.
2. $M = 0^*1 \cap P(r, 0^*1L)$ is regular.
3. $M = \{x \mid \exists y(r(|x|, |y|) \ \& \ x \in 0^*1 \ \& \ y \in L)\}$
4. Since M is regular, $\{|x| \mid x \in M\}$ is u.p.

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4. Since M is regular, $\{|x| \mid x \in M\}$ is u.p.
5. $r^{-1}(A) = \{|x| \mid x \in M\} \cup \{0\}$ is u.p.

Characterization as u.p. preserving relations

Theorem 1

A relation is regularity preserving iff it is u.p. preserving

Proof ($\leftarrow$)

Assume r is u.p. preserving. Suppose L is regular. From the MN relation for L, we get finitely many regular sets $L_1, ..., L_k$. Define R_i such that $\{y \mid xy \in L \ \& \ x \in L_i\}$

1. $\Sigma^* = L_1 \cup L_2 \cup ... \cup L_k$
2. $L = L \cap (L_1 \cup L_2 \cup ... \cup L_k)$

Characterization as u.p. preserving relations

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1. $\Sigma^* = L_1 \cup L_2 \cup ... \cup L_k$
2. $L = L \cap (L_1 \cup L_2 \cup ... \cup L_k)$
3. $P(r, L) = (P(r, L) \cap L_1) \cup (P(r, L) \cap L_2) \cup ... \cup (P(r, L) \cap L_k)$

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3. $P(r, L) = (P(r, L) \cap L_1) \cup (P(r, L) \cap L_2) \cup ... \cup (P(r, L) \cap L_k)$
4. $P(r, L) \cap L_i = \{x \mid \exists y(r(|x|, |y|) \ \& \ xy \in L\} \cap L_i$
5. $P(r, L) \cap L_i = \{x \mid |x| \in r^{-1}(\{|y| \mid y \in R_i\}\} \cap L_i$

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U.p. degenerating relations

Definition

If a binary relation r is U.P. degenerating then, $r^{-1}(A)$ is **finite** if A is **finite** and $r^{-1}(A)$ is **cofinite** if A is **infinite**

Note :- A is a Up set in above definition

U.p. degenerating relations

Examples of degenerating relations

Consider relation $r = \{(n, n) | n \in N\}$

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pairs which will be in relations r = { (1,1), (2,2), (3,3), (4,4), (5,5), ...}

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A is a range of relation r, i.e all possible outputs of relation r

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Is set A a up set? If it is a U.P set then is the relation r U.P. degenerating relation?

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Examples of degenerating relations

Consider relation $r = \{(n, n) | n \in N\}$

pairs which will be in relations r = { (1,1), (2,2), (3,3), (4,4), (5,5), ...}

Set A is a UP set

For each i, it can be seen that $r(i, j)$ holds for every j such that $i = j$

It can be seen that set A is infinite set and also it can be seen that $r^{-1}(A)$ is also cofinite.

It is also easy to argue that if set A would have been finite then by the condition on i and j, $r^{-1}(A)$ would also be finite

U.p. degenerating relations

Examples of degenerating relations

Consider relation $r = \{(\lfloor n^{\frac{1}{2}} \rfloor, n - \lfloor n^{\frac{1}{2}} \rfloor) | n \in N\}$

pairs which will be in relations r = { .....(3,12), (4,12) , (4,13), (4,14), (4,15) .... (4,20), (5,20), (5,21) .. (5,30), (6,30)..}

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$r^{-1}(A)$ is the domain of the relation r, i.e all possible inputs of relation r

Is set A a up set? Is relation r UP degenerating relation?

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Set A is a UP set

For each i, it can be seen that $r(i, j)$ holds for every j with $i^2 - i \leq j \leq i^2 + i$

Lets take $i = 4$, range for j will be $12 \leq j \leq 20$

It can be seen that set A is infinite set and also it can be seen that $r^{-1}(A)$ is also cofinite.

It is also easy to argue that if set A would have been finite then by the condition on i and j, $r^{-1}(A)$ would also be finite

U.p. degenerating relations

Theorem 2

If $r1$ and $r2$ are U.P. degenerating relations, then $P(r1, L) - P(r2, L)$ is finite for every regular Language L.

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Proof

Let L be any regular language over $\sum$

In proving Theorem 1, we found that regular sets $L_1, L_2, ..., L_k$ and $R_1, R_2, ..., R_k$ over $\sum$ such that, for any relation r on N

$P(r, L) = (P(r, L) \cap L_1) \cup (P(r, L) \cap L_2) \cup ... \cup (P(r, L) \cap L_k)$ .. (defined earlier)

$P(r, L) = \cup_{i=1}^{k}(\{x \in \sum^* \mid |x| \in r^{-1}(\{|y|, y \in R_i\})\} \cap L_i)$

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If $r1$ and $r2$ are U.P. degenerating relations, then $P(r1, L) - P(r2, L)$ is finite for every regular Language L.

Proof

1. For $r1$ and $r2$ u.p degenerating relations, we define
2. $L_{i,j} = \{x \in \sum^* ||x| \in r_j^{-1}(|y|, y \in R_i\})\}$ .. $(1 \leq i \leq k, 1 \leq j \leq 2)$

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3. $P(r1, L) - P(r2, L) = \cup_{i=1}^{k}(L_{i,1} \cap L_i) - \cup_{i=1}^{k}(L_{i,2} \cap L_i)$

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4. $P(r1, L) - P(r2, L) = \cup_{i=1}^{k}((L_{i,1} - L_{i,2}) \cap L_i)$

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4. $P(r1, L) - P(r2, L) = \cup_{i=1}^{k}((L_{i,1} - L_{i,2}) \cap L_i)$
5. $P(r1, L) - P(r2, L) = \cup_{i=1}^{k}((L_{i,1} \cap (L_{i,2})^c) \cap L_i)$

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5. $P(r1, L) - P(r2, L) = \cup_{i=1}^{k}((L_{i,1} \cap (L_{i,2})^c) \cap L_i)$
6. Now if R_i is finite, then $L_{i,1}$ will be finite and $(L_{i,2})^c$ will be co-finite and intersection of these two will be finite. Similarly we can observe the case when R_i will be infinite.

U.p. degenerating relations

Examples

Examples of degenerating relations are
$r1 = \{(\lfloor n^{\frac{1}{2}} \rfloor, n - \lfloor n^{\frac{1}{2}} \rfloor) | n \in N\}$
$r2 = \{(\lfloor n^{\frac{1}{3}} \rfloor, n - \lfloor n^{\frac{1}{3}} \rfloor) | n \in N\}$
$r3 = \{(\lfloor \log_2(n) \rfloor, n - \lfloor \log_2(n) \rfloor) | n \in N\}$

If we take any of the the above two relations, lets say r_1 and r_3 and a regular language L then $P(r1, L) - P(r_3, L)$ will be finite.
For sufficiently long string x,
$x \in P(r1, L) \iff x \in P(r3, L) \iff x \in P(r2, L)$

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A function $f : N \to N$ is u.p. reducible if, for every modulus m, there exists a period p such that $f(n) \equiv f(n+p) \bmod m$ for all but finitely many $n \in N$.

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1. $f(n) = n^2$ is u.p. reducible.

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   1. For any m, $\exists p(n^2 \equiv (n+p)^2 \bmod m)$

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   1. For any m, $\exists p(n^2 \equiv (n+p)^2 \bmod m)$
   2. $(n+p)^2 \bmod m = (n^2 \bmod m + p(2n+p) \bmod m) \bmod m$

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    3. If we take $p = m$ then, $n^2 \equiv n^2 \bmod m$ which holds for all n.

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    3. If we take $p = m$ then, $n^2 \equiv n^2 \bmod m$ which holds for all n.
2. $$f(n) = \begin{cases} 0 \textit{ if } n \textit{ is even} \\ n! \textit{ if } n \textit{ is odd} \end{cases}$$

Definitions

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A function $f : N \to N$ is u.p. reducible if, for every modulus m, there exists a period p such that $f(n) \equiv f(n+p) \bmod m$ for all but finitely many $n \in N$.

1. $f(n) = n^2$ is u.p. reducible.
    1. For any m, $\exists p(n^2 \equiv (n+p)^2 \bmod m)$
    2. $(n+p)^2 \bmod m = (n^2 \bmod m + p(2n+p) \bmod m) \bmod m$
    3. If we take $p = m$ then, $n^2 \equiv n^2 \bmod m$ which holds for all n.
2. $$f(n) = \begin{cases} 0 \;\; if \;\; n \; is \; even \\ n! \;\; if \;\; n \; is \; odd \end{cases}$$
    1. u.p. reducible $\forall n \geq m$

Definitions

Definition 2

A function $f : N \to N$ is essentially increasing if, for every k, $f(n) \geq k$ for all but finitely many $n \in N$.

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A function $f : N \rightarrow N$ is essentially increasing if, for every k, $f(n) \geq k$ for all but finitely many $n \in N$.

1. $f(n) = n$
    1. For any and every k, $f(n) \geq k$ for $n \geq k$

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   1. For any and every k, $f(n) \geq k$ for $n \geq k$
2. $$f(n) = \begin{cases} 0 \ \mathit{if} \ \ n \ \mathit{is} \ \mathit{even} \\ n! \ \mathit{if} \ \ n \ \mathit{is} \ \mathit{odd} \end{cases}$$
   1. Not essentially increasing

Definitions

Definition 1

$F = \{f \mid f \text{ is essentially increasing and u.p. reducible}\}$

Definitions

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$F = \{f \mid f \text{ is essentially increasing and u.p. reducible}\}$

Definition 2

$G = \{f \mid f \text{ is u.p. preserving}\}$

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$F = \{f \mid f \text{ is essentially increasing and u.p. reducible}\}$

Definition 2

$G = \{f \mid f \text{ is u.p. preserving}\}$

Definition 3

$H = \{f \mid f \text{ is u.p. reducible}\}$

Class of Regularity Preserving functions

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    2. $\Rightarrow (f(n) \in A \Leftrightarrow f(n+p) \in A) \Rightarrow (n \in f^{-1}(A) \Leftrightarrow n+p \in f^{-1}(A))$
    3. $f^{-1}(A)$ is u.p.

Proof ($\neq$)

Recall a $f \notin F$. However, $f^{-1}(A)$ is u.p. so $f \in G$

Class of Regularity Preserving functions

Theorem 3

$F \subset G$

1. $f(n) = n$. $f \in F \Rightarrow f \in G$. Therefore, f is u.p. preserving. Using Theorem 1, f is also regularity preserving.

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2. $f(n) = n^2 - n$?

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Assume $f \in G$. Let m be any positive integer.

1. $f^{-1}(j \mid j \equiv n \; mod \; m)$ is u.p. for $0 \leq n \leq m-1$
2. We have m u.p. sets. If p is a common period for these sets, $f(n) \equiv f(n+p)$ mod m for all but finitely many n.

Class of Regularity Preserving functions

Theorem 4

$G \subset H$

Proof ($\neq$)

1. Let A be a set which is not u.p. Let

$$f(n) = \begin{cases} 0 \ \textit{if} \ \ n \ \in \ A \\ n! \ \textit{if} \ \ n \ \notin \ A \end{cases}$$

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$$f(n) = \begin{cases} 0 \;\; if \;\; n \;\; \in \;\; A \\ n! \;\; if \;\; n \;\; \notin \;\; A \end{cases}$$

2. $\forall m$ and $\forall n \geq m$, $f(n) \equiv f(n+1) \equiv 0 \mod m$.
3. $f \in H$. $f^{-1}(\{0\}) = A$. So, $f \notin G$

Class of Regularity Functions

Theorem 5

$H = \{f | P(f, L) \text{ is regular for every regular } L \subseteq \{1\}^*\}$

Proof $\supseteq$

Let f be a function. Define an essentially increasing function $g(n) = n + f(n)$

- Let us assume $P(f, L)$ is regular for every regular $L \subseteq \{1\}^*$. Suppose A is U.P.

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- Now we can observe that A and $g^{-1}(A)$ both are u.p sets, therefore g is u.p preserving and thus $g \in G \subset H$ from theorem 4

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- Now, $f(n) = g(n) - n$, therefore $f \in H$

Class of Regularity Functions

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$H = \{f | P(f, L)$ is regular for every regular $L \subseteq \{1\}^*\}$

Proof $\subseteq$

Assume f is u.p reducible and g be an essentially increasing function $g(n) = n + f(n)$

- Now $g \in F$ as it is always increasing and u.p reducible.

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Assume f is u.p reducible and g be an essentially increasing function $g(n) = n + f(n)$

- Now $g \in F$ as it is always increasing and u.p reducible.
- g is also u.p. preserving by Theorem 3
  Suppose $L \subseteq \{1\}^*$ is regular.

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  Suppose $L \subseteq \{1\}^*$ is regular.
- $P(f, L) = \{x \mid \exists y(f(|x|) = |y| \;\&\; xy \in L)\}$

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- $P(f, L) = \{x \mid \exists y(g(|x|) = |xy| \; \& \; xy \in L)\}$ .. $(g(n) = n + f(n))$
- $P(f, L) = \{x \; \in \{1\}^* | \; |x| = g^{-1}(\{|z| | z \in L\})\}$
  Now $P(f, L)$ will be regular by Lemma 2

Class of Regularity Functions

Theorem 6

$G = \{f \in H \mid f^{-1}(\{j\}) \text{ is u.p for every } j\}$

Proof ($\subseteq$)

Assume $f \in G$. By Theorem 4, $f \in H$.
Since $\{j\}$ is certainly u.p. for each j, $f^{-1}(\{j\})$ is u.p. (By definition of u.p. preserving functions)

Class of Regularity Functions

Proof ($\supseteq$)

Assume $f \in H$ and $f^{-1}(\{j\})$ is u.p for every j. Suppose A is u.p with n_o, p such that
$(\forall n \geq n_o)(n \in A \Leftrightarrow n + p \in A)$

$\{j \in A \mid j \geq n_o\} = (\bigcup_{n \in A, n_o \leq n < n_o + p} \{j \mid j \equiv (n \bmod p)\}) - \{j \mid j < n_o\}$

$f^{-1}(A) = f^{-1}(\{j \in A | j < n_0\}) \cup f^{-1}(\{j \in A | j \geq n_0\})$
$f^{-1}(A) = (\bigcup_{j \in A, j < n_0} f^{-1}(\{j\})) \cup ((\bigcup_{n \in A, n_o \leq n < n_o + p} \{i | f(i) \equiv (n \bmod p)\}) - (\bigcup_{j < n_0} f^{-1}(\{j\})))$

Since $f \in H$, each of the sets $\{i \mid f(i) \equiv n \; mod \; p\}$ is u.p. From the assumption, $f^{-1}(\{j\})$ is u.p for every j. So, $f^{-1}(A)$ is u.p.

Contents

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3. Characterization as u.p. preserving relations
4. U.p. degenerating relations
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Applications

1. Given a binary relation, we can characterize it as a regularity preserving relation. If we have a regular language, we know for sure that a finite state automaton exists for the set of strings that satisfy the given relation.
2. Can achieve performance enhancements on pattern matching.

References

J. I. Sieferas and R McNaughton.
Regularity-preserving relations.