

Parikh's Theorem for CFL's

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Outline

- 1 Parikh map
- 2 Parikh's theorem
- 3 Proof

Rohit Parikh



Parikh map of a string

- Let $A = \{a_1, \dots, a_n\}$ be a finite alphabet.
- The **Parikh map** of a string $w \in A^*$ is a vector in \mathbb{N}^n given by:

$$\psi(w) = (\#_{a_1}(w), \#_{a_2}(w), \dots, \#_{a_n}(w)).$$

- For example if $A = \{a, b\}$, then $\psi(baabb) = (2, 3)$.
- Parikh map is also called the “letter-count” of a string.
- Extend the map to languages L over A :

$$\psi(L) = \{\psi(w) \mid w \in L\}.$$

- What is $\psi(\{a^n b^n \mid n \geq 0\})$?

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 - $\{(m, n) \mid m \leq n\}$.

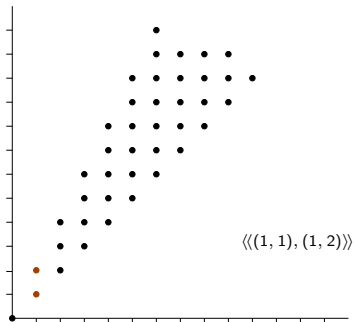
Semi-linear sets of vectors

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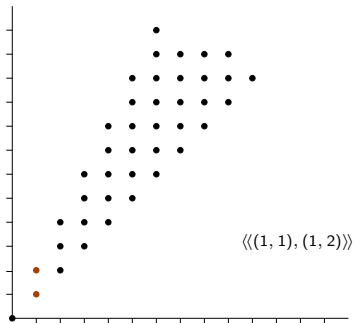
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- A set of vectors is called **semi-linear** if it is a finite union of linear sets.

Parikh's Theorem for CFL's

Theorem (Parikh)

The Parikh map of a CFL is a semi-linear set. That is, if L is a CFL then $\psi(L)$ is semi-linear.

Some corollaries:

- Every CFL is “letter-equivalent” to a regular language.
 - For example: $\psi(\{a^n b^n \mid n \geq 0\})$

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Is Parikh's theorem a sufficient condition for context-freeness as well?

- No, since $\psi(\{a^n b^n c^n \mid n \geq 0\}) = \{(n, n, n) \mid n \geq 0\}$ is semi-linear.

Running example

CFG G_1

$$S \rightarrow XC \mid AY$$

$$X \rightarrow aXb \mid ab$$

$$Y \rightarrow bYc \mid bc$$

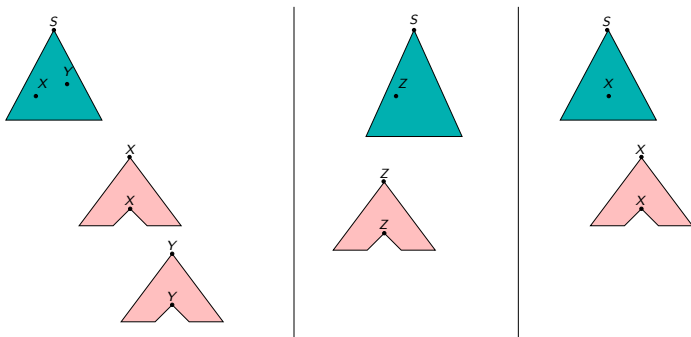
$$A \rightarrow aA \mid a$$

$$C \rightarrow cC \mid c$$

- What is the language generated?
- What is the Parikh image of this language?
- Write it as a semi-linear set.

Idea of proof

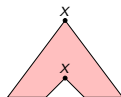
- Group parse trees of G into a finite number of buckets
- Each bucket is represented by a “minimal” parse tree and associated “basic pumps”.
- Argue that the set of strings derived in each bucket gives rise to a linear set.



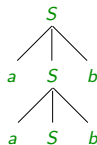
Proof: Pumps

Let us fix a CFG $G = (N, A, S, P)$ in CNF form.

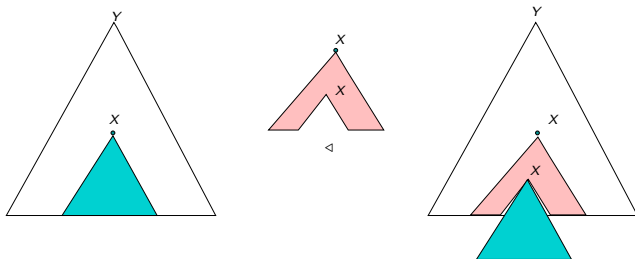
- A **pump** is a derivation tree s which has at least two nodes, and $yield(s) = x \cdot root(s) \cdot y$, for some terminal strings x, y .



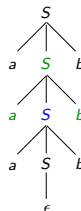
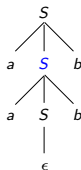
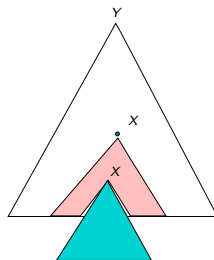
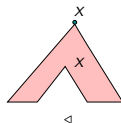
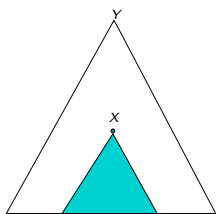
- Example pumps for grammar $S \rightarrow aSb \mid SS \mid \epsilon$:



Growing and shrinking with pumps

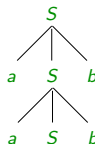


Growing and shrinking with pumps



Basic Pumps

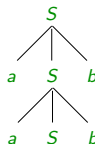
A pump is **basic** if it is \Leftarrow -minimal. Thus a pump s is a basic pump if it cannot be shrunk by some pump and still remain a pump.



- First pump is basic but second is not.
- How many pumps are there?

Basic Pumps

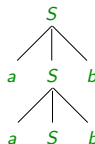
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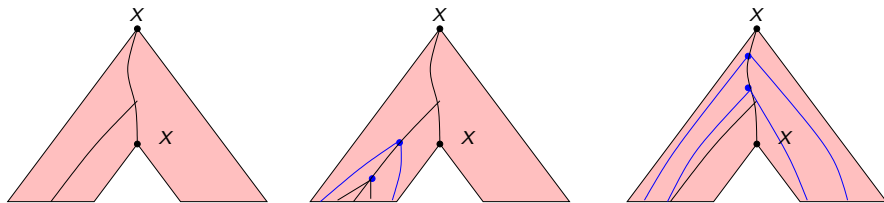
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- First pump is basic but second is not.
- How many pumps are there? Infinitely many.
- How many basic pumps are there? Finitely many since their height is bounded by $2|N|$ (See argument on next slide).
- Let p be the number of basic pumps.

Basic pumps height bounded by $2N$

Consider the longest path from root to leaf in a basic pump.
The number of nodes on it is bounded by $2|N| + 1$.



\leq relation on parse trees

- Let s and t be derivation trees of terminal strings starting from start symbol S .
- Then we say $s \leq t$ iff t can be grown from s by basic pumps whose non-terminals are contained in those of s (thus the pumps do not introduce any new non-terminals, and s and t have the same set of non-terminal nodes).
- A parse tree s is thus \leq -minimal if it does not contain a basic pump that can be cut out **without** reducing the set of non-terminals that occur in s .
- \leq -minimal trees can be seen to be finite in number: their height is bounded by $(p + 1)(|N| + 1)$.
- Ex: What are the \leq -minimal parse trees for grammar G_1 ?

Overall strategy of Proof

- Begin with the \leq -minimal derivation trees, say s_1, \dots, s_k .
- Associate with each s_i the set of basic pumps whose non-terminals are **contained in** that of s_i .
- Argue that the set of derivation trees obtained by starting with s_i and growing using the associated basic pumps, (let us call this the “bucket” of parse trees associated with s_i) gives rise to a set of strings whose Parikh map is a linear set.

Exercise

Describe the “buckets” for G_1 , and the corresponding linear sets.