Algebraic Approach to Automata Theory

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Outline

Algebraic approach to automata: Overview

An algebra is a set with some operations on it, satisfying some properties.
E.g. a group is a nonempty set with a unit 1, a binary operation satisfying associativity $(x \circ (y \circ z) = (x \circ y) \circ z)$ and a unary operation satisfying an inverse property $(x \circ inv(x) = 1)$

- Define language recognition via morphisms into a monoid
- Analogous result to canonical automaton (Myhill-Nerode)
- Helps in characterising class of FO-definable languages
- Helps in automaton constructions

Monoids

- A monoid is an algebra $(M, \circ, 1)$, where
  - M is a set containing the element 1,
  - $\circ$ is an associative binary operation on M, and
  - 1 is the unit element (sometimes called identity) with respect to $\circ$, so for every $m \in M$, $m \circ 1 = m = 1 \circ m$
  - Sometimes a monoid may have a zero element 0 (if so, unique), such that for every $m \in M$, $m \circ 0 = 0 = 0 \circ m$
- Examples: $(\mathbb{N}, \times)$, $(\mathbb{N}^{\geq 1}, \times)$, $(\mathbb{N}, +)$, $(A^*, \cdot)$. First three are commutative monoids (operation is commutative), first one has a zero, others do not
- Nonexample: $(\mathbb{N}^{\geq 1}, +) = \{1, 2, 3, \dots\}$ with addition (why?)

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- Nonexample: $(\mathbb{N}^{\geq 1}, +) = \{1, 2, 3, \dots\}$ with addition (why?)
- More noncommutative examples: $Fun(X) = (X \to X, \circ, id)$, $Rel(X) = (2^{X \times X}, \circ, id)$ where
  - $X \to X$ is the set of all functions from a set X to itself
  - $2^{X \times X}$ is the set of subsets of $X \times X$, that is, all relations on X
  - $f \circ g$ is function composition, $R \circ S$ is relation composition, both are not commutative

Monoid morphisms

- A morphism from a monoid $(M, \circ_M, 1_M)$ to a monoid $(N, \circ_N, 1_N)$ is a map $h : M \to N$, satisfying
  - $h(1_M) = 1_N$, and,
  - $h(m \circ_M m') = h(m) \circ_N h(m')$.
- Example: $h : A^* \to \mathbb{N}$, given by

$$h(w) = |w|$$

is a morphism from $(A^*, \cdot, \epsilon)$ to $(\mathbb{N}, +, 0)$.

Language recognition via monoid morphisms

- A language $L \subseteq A^*$ is said to be recognized by monoid $(M, \circ, 1)$ if there is a morphism h from $(A^*, \cdot, \epsilon)$ to $(M, \circ, 1)$, and a recognizing subset X of M such that

$$L = h^{-1}(X)$$

  Exercise: Show that $L = h^{-1}(h(L))$.
- If L is recognized by some finite M, we say that L is recognizable
- Finite monoids are finitely generated, that is, there is a finite set of elements (which we can take to be the alphabet) such that every element is a multiplication described by words over the alphabet

Example of language recognition via monoid morphisms

Monoid $M = (\{1, m\}, \circ, 1)$ generated by m
Multiplication table shown

$\circ$	1	m
1	1	m
m	m	m

- Consider $h : A^* \to M$ given by $h(w) = m$ iff $w \neq \epsilon$ (why morphism?)
- Then M recognizes A^+, since $h^{-1}(\{m\}) = A^+$.
- M also recognizes $\{\epsilon\}$ (taking $X = \{1\}$), A^* (taking $X = \{1, m\}$), and $\emptyset$ (taking $X = \{\}$).

Exercise

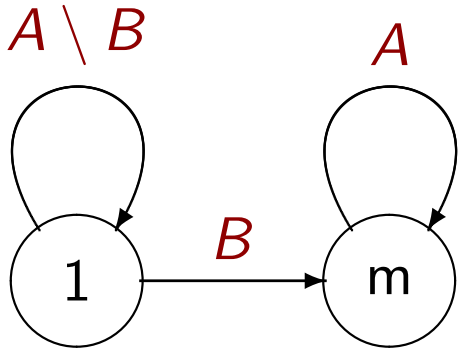
Show that the language of words with an odd number of a's over the alphabet $A = \{a, b\}$ is recognizable.

Finitely generated monoid to deterministic automaton

M generated by m

$\circ$	1	m
1	1	m
m	m	m

Let $B = h^{-1}(m) \cap A$, $h(A \setminus B) = 1$



- Let $L \subseteq A^*$ a language recognized by a monoid $(M, \circ, 1)$.
- That is, a morphism $h : A^* \to M$ and $X \subseteq M$ and the morphism maps subset L to subset X, $L = h^{-1}(X)$.
- Define a DA $\mathcal{A}(M) = (M, \delta, 1, X)$, where

$$\delta(m, a) = m \circ h(a).$$

- To prove that: $\mathcal{A}(M)$ accepts L.

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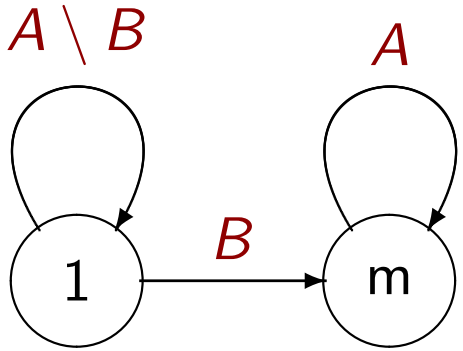

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- Lemma: $\widehat{\delta}(1, w) = h(w)$. Proof by induction on $|w|$.

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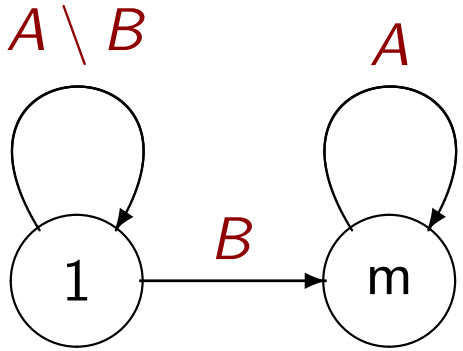

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- To prove that: $\mathcal{A}(M)$ accepts L.
- Lemma: $\widehat{\delta}(1, w) = h(w)$. Proof by induction on $|w|$.
- Main proof: $\mathcal{A}(M)$ accepts exactly the w such that $h(w) \in X$. So $\mathcal{A}(M)$ accepts $h^{-1}(X) = L$.

Transition Monoid of a DA

Let $\mathcal{A} = (Q, \delta, s)$ be a deterministic transition system.

- For $w \in A^*$, define $f_w : Q \to Q$ by

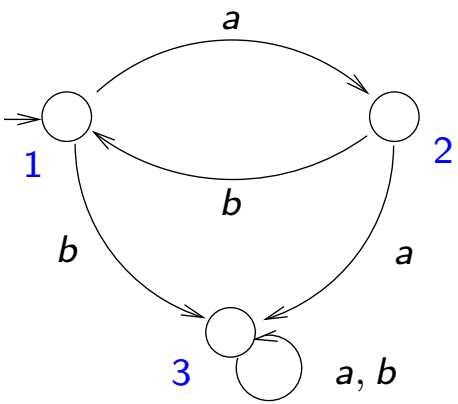
$$f_w(q) = \widehat{\delta}(q, w).$$

- Consider the monoid

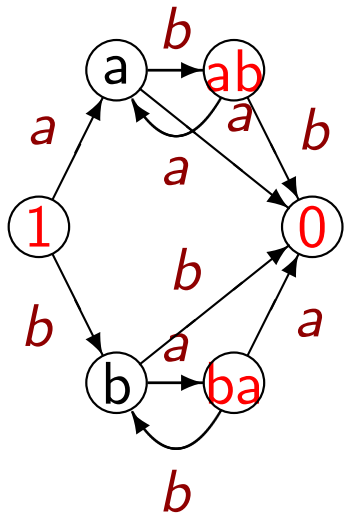
$$M(\mathcal{A}) = (\{f_w \mid w \in A^*\}, \circ, f_\epsilon).$$

  with the morphism $h(w) = f_w$.
- $M(\mathcal{A})$ is a submonoid of $Fun(Q) = (Q \to Q, \circ, id)$, it is called the transition monoid of $\mathcal{A}$.

Example: DFA to Transition Monoid to DFA



∘	1	a	b	ab	ba	0
1	1	a	b	ab	ba	0
a	a	0	ab	0	a	0
b	b	ba	0	b	0	0
ab	ab	a	0	ab	0	0
ba	ba	0	b	0	ba	0
0	0	0	0	0	0	0



Distinct elements of $M(\mathcal{A})$ are
$\{f_\epsilon = 1, f_a, f_b, f_{ab}, f_{ba}, f_{aa} = f_{bb} = 0\}$.
Multiplication table as shown. Idempotents e satisfy $ee = e$.
Right-multiplication by generators a, b is coloured.

Write f_a as $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$, or (233). $f_b = (313), f_{ab} = (133)$, $f_{ba} = (323), 0 = (333)$.

Question: If Q is finite, how many elements can $M(\mathcal{A})$ have?

Homework

Problem (McNaughton-Papert 1971)

Find transition monoids for two DFA with states $\{s, p, q\}$, initial state s. Also construct the automata from the monoids and compare what you get.

1. $\delta_1 = s \xrightarrow{a} p, p \xrightarrow{a} s, q \xrightarrow{a} q,\ s \xrightarrow{b} s, p \xrightarrow{b} q, q \xrightarrow{b} p$
2. $\delta_2 = s \xrightarrow{a} p, p \xrightarrow{a} q, q \xrightarrow{a} q,\ s \xrightarrow{b} s, p \xrightarrow{b} s, q \xrightarrow{b} p$

- Two automata look similar
- First monoid has 6 elements, second monoid has 8 elements. Using them can say something about languages they recognize
- There is a language recognized by first monoid which is not definable in first-order logic with $<$. All languages recognized by second one are defined by first-order logic sentences

Syntactic Monoid of a language

- Let $\mathcal{A}_{\equiv_L} = (Q, s, \delta, F)$ be the canonical automaton for a language $L \subseteq A^*$.
- The transition monoid of $\mathcal{A}_{\equiv_L}$ is called the syntactic monoid of L, we write $M(L)$ rather than $M(\mathcal{A}_{\equiv_L})$.
- The surjective morphism $h_L : A^* \to M(L)$ is called the syntactic morphism of L.

Syntactic Congruence of a language

- Define an equivalence relation $\cong_L$ on A^*, induced by L, as

$$u \cong_L v \text{ iff } \forall x, y \in A^* : \ xuy \in L \text{ iff } xvy \in L.$$

- $\cong_L$ is called the syntactic congruence of L.
- Check that $\cong_L$ is a two-sided congruence:
  - That is, $\cong_L$ is both a left-congruence (i.e $u \cong_L v$ implies $wu \cong_L wv$, for each $w \in A^*$) and a right-congruence (i.e $u \cong_L v$ implies $uw \cong_L vw$).
  - Equivalently, $u \cong_L u'$ and $v \cong_L v'$ imples $uv \cong_L u'v'$.
- $\cong_L$ refines the canonical MN relation, $\equiv_L$, for L.
- Example?

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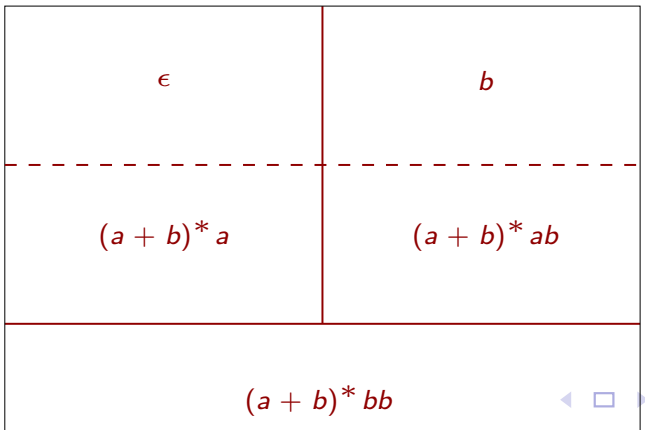
- Example? Consider the language $(a+b)^*bb$:



Characterization of the syntactic monoid

Claim

For a canonical DA $\mathcal{A} = (Q, \delta, s, F)$,

$$f_u = f_v \text{ iff } u \cong_L v.$$

Proof: ($\Longrightarrow$) By definition the element f_w of the transition monoid of the canonical DA is $\widehat{\delta}(\_, w)$.

$xuy \in L$ iff $(f_x \circ f_u \circ f_y)(s) \in F$ in the canonical DA
(in other words, $\widehat{\delta}(s, xuy) \in F$) in the canonical DA
iff $(f_x \circ f_v \circ f_y)(s) \in F$ in the canonical DA
(in other words, $\widehat{\delta}(s, xvy) \in F$) in the canonical DA
iff $xvy \in L$

Exercise: Prove the ($\Longleftarrow$) direction.

Thus the syntactic congruence $u \cong_L v$ matches the equality $f_u = f_v$ derived from the canonical DA.

Syntactic monoid via syntactic congruence

For a language $L \subseteq A^*$, consider the monoid $A^*/\cong_L$:

- elements are congruence classes under $\cong_L$,
- take the unit to be $[\epsilon]$,
- multiply two congruences by defining:

$$[u] \circ [v] = [uv].$$

Claim

The monoids $M(L)$ and $A^*/\cong_L$ are isomorphic.

(Use the morphism $f_w \mapsto [w]$.)

Algebraic definition of regular languages

Theorem (Myhill-Nerode)

Let $L \subseteq A^*$. Then the following are equivalent:

1. *L is regular*
2. *The syntactic monoid of L, i.e. $M(L)$, is finite*
3. *L is recognizable*

Proof:

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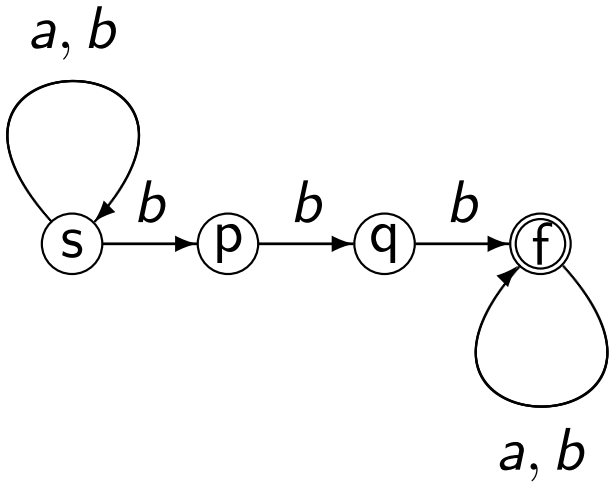
(2) $\implies$ (3): Syntactic morphism $h_L : A^* \to M(L)$, given by $w \mapsto f_w$, recognizes L.

(3) $\implies$ (1): Given finite M recognizing L, finite $\mathcal{A}(M)$ accepts L.

Corollary (Rabin-Scott, Shepherdson-Sturgis)

Nondeterministic and two-way automata accept regular languages

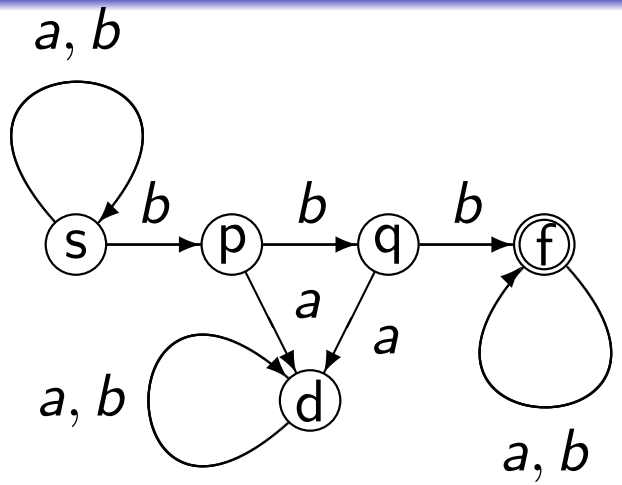
NFA to monoid of relations



$$\Delta(a) = \{(s,s),(f,f)\}$$
$$\Delta(b) = \{(s,s),(s,p),(p,q),(q,f),(f,f)\}$$
$$\widehat{\Delta}(\epsilon) = id = \{(s,s),(p,p),(q,q),(f,f)\}$$
$$\widehat{\Delta}(bb) = \{(s,s),(s,p),(s,q),(p,f),(f,f)\}$$
$$\widehat{\Delta}(bbb) = \{(s,s),(s,p),(s,q),(s,f),(f,f)\}$$
$$\widehat{\Delta}(ab) = \{(s,s),(s,p),(f,f)\}$$
$$\widehat{\Delta}(abb) = \{(s,s),(s,p),(s,q),(f,f)\}$$

- Think of (Q,Δ), where every letter of the alphabet defines a transition relation $\Delta : A \to 2^{Q\times Q}$
- $\widehat{\Delta}$ as mapping a word to a relation on states $A^* \to 2^{Q\times Q}$
- $\widehat{\Delta}(\epsilon) = id, \quad \widehat{\Delta}(wa) = \widehat{\Delta}(w) \circ \Delta(a)$
- That is, $\widehat{\Delta}$ is a morphism into monoid of relations $Rel(Q)$

Rabin-Scott construction as a monoid construction



∘	a	b	ab	bb	abb	bbb
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b	a	bb	b	bbb	bb	bbb
ab	a	abb	b	bbb	bb	bbb
bb	a	bbb	b	bbb	bb	bbb
abb	a	bbb	b	bbb	bb	bbb
bbb	a	bbb	b	bbb	bb	bbb

- Complete the NFA to make it total, then take $Rel(Q)$
- The monoid has 8 elements $\{1, a, b, ab, bb, abb, bbb, 0\}$
- 1,0 are not shown in the multiplication table
- Follow abstractly how the monoid is calculated

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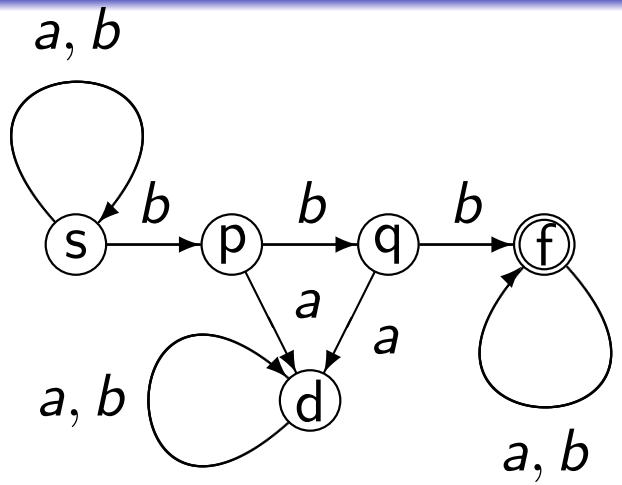

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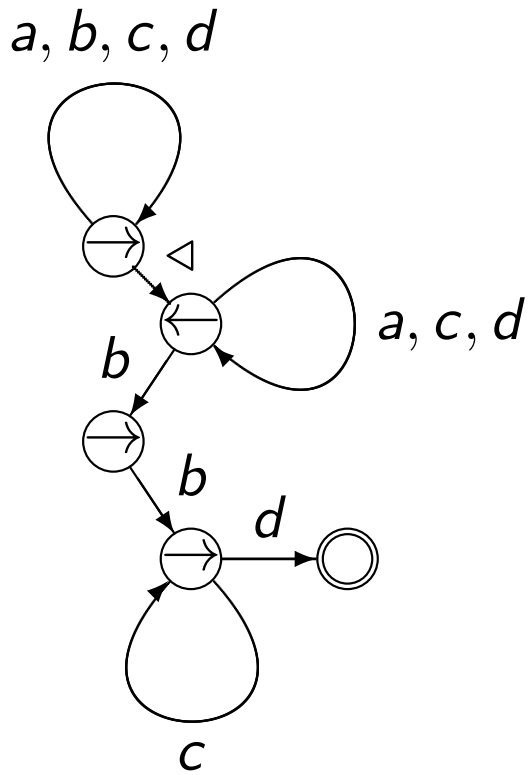
- Follow abstractly how the monoid is calculated
- Exercise: Which monoid elements correspond to initial and final states of the automaton?

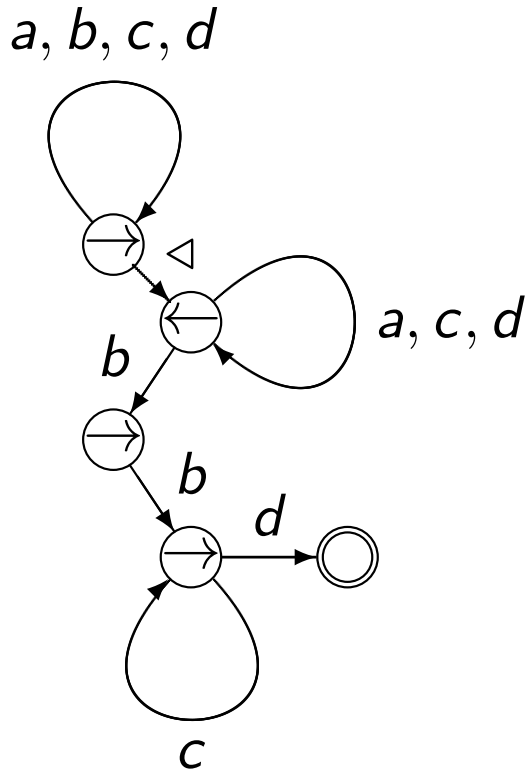
Shepherdson construction for two-way automata

- Left and right endmarkers $\triangleright w \triangleleft$ in alphabet A for input word w
- Q is union of left-moving states $\overleftarrow{Q}$ and right-moving states $\overrightarrow{Q}$
- Transition $\delta : \overleftarrow{Q} \times (A \setminus \{\triangleright\}) \to Q$ reads letter and moves left, $\delta : \overrightarrow{Q} \times (A \setminus \{\triangleleft\}) \to Q$ reads letter and moves right as usual
- Partial functions (2DFA): moving left of left endmarker and moving right of right endmarker are undefined



Birget monoids for two-way automata

- Have to consider four types of morphic extensions for δ:
  - Either $h(\rightarrow w \rightarrow) : \overrightarrow{Q} \rightarrow \overrightarrow{Q}$ enters from left exits from right
  - Or $h(\overrightarrow{\leftarrow} w) : \overrightarrow{Q} \rightarrow \overleftarrow{Q}$ enters from left exits from left
  - Either $h(\leftarrow w \leftarrow) : \overleftarrow{Q} \rightarrow \overleftarrow{Q}$ enters from right exits from left
  - Or $h(w \overleftarrow{\rightarrow}) : \overleftarrow{Q} \rightarrow \overrightarrow{Q}$ enters from right exits from right



Problem

Construct a monoid recognizing the language accepted by a 2DFA

(Warning: there are missing cases above)