Undecidability of the Halting Problem

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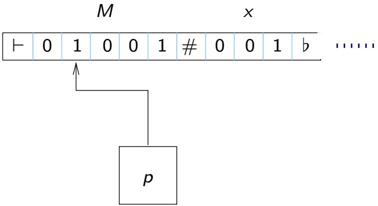
16 November 2021

Outline

Universal Turing machine



- We can construct a TM U that takes the encoding of a TM M and its input x, and "interprets" M on the input x.
- U accepts if M accepts x, rejects if M rejects x, and loops if M loops on x.

Encoding a TM as a $\{0,1\}$-string

$$0^n10^m10^k10^s10^t10^r10^u10^v\,1\,0^p10^a10^q10^b10\,1\,0^{p'}10^{a'}10^{q'}10^{b'}100\cdots1\,0^{p''}10^{a''}10^{q''}10^{b''}10.$$

represents a TM M with

- states $\{1,2,\ldots,n\}$.
- Tape alphabet $\{1,2,\ldots,m\}$.
- Input alphabet $\{1,2,\ldots,k\}$ (with $k<m$).
- Start state $s\in\{1,2,\ldots,n\}$.
- Accept state $t\in\{1,2,\ldots,n\}$.
- Reject state $r\in\{1,2,\ldots,n\}$.
- Left-end marker symbol $u\in\{k+1,\ldots,m\}$.
- Blank symbol $v\in\{k+1,\ldots,m\}$.
- Each string $0^p10^a10^q10^b10$ represents the transition $(p,a)\to(q,b,L)$.

Example encoding of TM and its input

Input is encoded as $0^a10^b10^c$ etc.

Exercise: What does the following TM do on input 001010?

Example encoding of a TM

0001000010010100100010001000 1 01000101000100 1 0100100100100 1 010101010.

[Assume accept and reject states are sink states]

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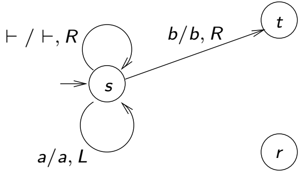
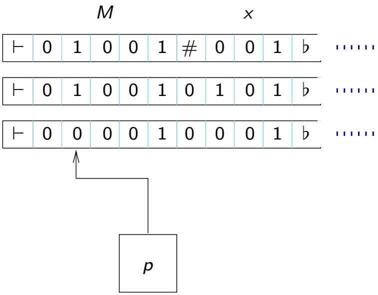


How the Universal Turing machine works



- Use 3 tapes: for input $M\#x$, for current configuration, and for current state and position of head.
- Repeat:
  - Execute the transition of M applicable in the current config.
- Accept if M gets into t state, Reject if M gets into r state.

Halting Problem for Turing machines

- Fix an encoding *enc* of TMs as above.
- Define the language

$$\mathrm{HP} = \{enc(M)\#enc(x) \mid M \text{ halts on } x\}.$$

- What can we say about the language HP?
  - Is recursively enumerable, since we can use the Universal TM to accept it.

Undecidability of HP

Theorem (Turing 1936)

The language HP *is not recursive.*

Proving undecidability of HP

Assume that we have a Turing machine M which decides HP. Then we can compute each entry of the table below:

	ϵ	0	1	00	01	10	11	000	001	010	011	111	$\cdots$
M_ϵ	L	H	L	L	L	H	H	L	L	L	L	L	$\cdots$
M_0	L	L	L	L	L	L	L	L	L	L	L	L	$\cdots$
M_1	H	H	L	H	L	H	H	L	L	H	L	H	$\cdots$
M_{00}	L	L	L	L	L	L	L	L	L	L	L	L	$\cdots$
M_{01}	L	H	L	L	L	H	H	L	L	L	L	L	$\cdots$
M_{10}	H	H	L	H	L	H	H	L	L	H	L	H	$\cdots$
M_{11}	L	H	L	L	L	H	H	L	L	L	L	L	$\cdots$
M_{000}	L	L	L	L	L	L	H	L	L	L	H	L	$\cdots$
$\vdots$													

- For each $x \in \{0,1\}^*$ let M_x denote the TM
  - M, if x is the encoding of TM M with input alphabet $\{0,1\}$.
  - M_{loop} otherwise, where M_{loop} is a one-state Turing machine that loops on all its inputs.
- Table entry (x, y) tells whether TM M_x halts on the input y. Note that y is an (unencoded) input in $\{0,1\}^*$.

A TM N that behaves differently from all TM's

- Let us assume we have a TM M that decides HP.
- Then we can define a TM N as follows: Given input $x \in \{0,1\}^*$, it
  - runs as M on $x\#enc(x)$.
  - If M accepts (i.e. M_x halts on x), goes to a new "looping" state l and loops there.
  - If M rejects (i.e. M_x loops on x), goes to the accept state t'.
- N essentially "complements the diagonal" of the table: Given input $x \in \{0,1\}^*$ it halts iff M_x loops on x.
- Consider $y = enc(N)$. Then y cannot occur as any row of the table since the behaviour of N differs from all rows in the table. This is a contradiction.

How N behaves

	ϵ	0	1	00	01	10	11	000	001	010	011	111	...
M_ϵ	L	H	L	L	L	H	H	L	L	L	L	L	...
M_0	L	L	L	L	L	L	L	L	L	L	L	L	...
M_1	H	H	L	H	L	H	H	L	L	H	L	H	...
M_{00}	L	L	L	L	L	L	L	L	L	L	L	L	...
M_{01}	L	H	L	L	L	H	H	L	L	L	L	L	...
M_{10}	H	H	L	H	L	H	H	L	L	H	L	H	...
M_{11}	L	H	L	L	L	H	H	L	L	L	L	L	...
M_{000}	L	L	L	L	L	L	H	L	L	L	H	L	...
$\vdots$													
N	H	H	H	H	H	L	L	H	...				
$\vdots$													

The constructed TM N complements the diagonal of the table, and hence does not occur as any of the TM's listed. This is not possible!

Complement of HP is not r.e.

Fact 1: If L and $\overline{L}$ are both r.e. then L (and $\overline{L}$) must be recursive.

- Let M accept L and M' accept $\overline{L}$.
- We can construct a total TM that simulates M and M' on given input, one step at a time.
- Accept if M accepts, Reject if M' accepts.

Fact 2: HP is recursively enumerable.

- Just run the universal TM U on input $M\#x$; accept iff U halts (i.e. M accepts or rejects x).

Corollary

The language $\neg\mathrm{HP}$ is not even recursively enumerable.

Where HP lies