Visibly Pushdown Automata

Deepak D'Souza

Department of Computer Science and Automation
Indian Institute of Science, Bangalore.

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Outline

1. Visibly Pushdown Automata
2. Closure properties of VPL
3. Determinization
4. Logical Characterization

Visibly Pushdown Automata

- A sub-class of Pushdown Automata (PDA's) in which pushing/popping from the stack is dictated by input letters.
- Useful properties for verification
  - Closed under operations like union, intersection, complementation, concatentation, Kleene-*.
  - Decidable language inclusion and universality problems.

Proposed by Rajeev Alur and P. Madhusudan in STOC 2004.

Example VPA

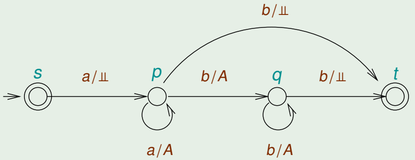
Example VPA for $\{a^n b^n \mid n \geq 0\}$

$$\Sigma_c = \{a\}$$
$$\Sigma_r = \{b\}$$
$$\Sigma_{int} = \emptyset$$

$$(s, a, p, \perp\!\!\!\perp)$$
$$(p, a, p, A)$$
$$(p, b, A, q)$$
$$(q, b, A, q)$$
$$(q, b, \perp\!\!\!\perp, t)$$

$$F = \{s, t\}.$$

State Diagram of VPA



Definitions

A VPA over a partitioned alphabet $\widetilde{\Sigma} = (\Sigma_c, \Sigma_r, \Sigma_{int})$ is a structure $M = (Q, Q_0, \Gamma, \perp, \delta, F)$ where Q is a finite set of states, Q_0 is a set of initial states, Γ is a stack alphabet with $\perp \in \Gamma$, F is a set of final states, and δ is the transition relation of the form:

- (p, a, q, A) if $a \in \Sigma_c$ (push transition)
- (p, a, A, q) if $a \in \Sigma_r$ (pop transition)
- (p, a, q) if $a \in \Sigma_{int}$ (internal transition).

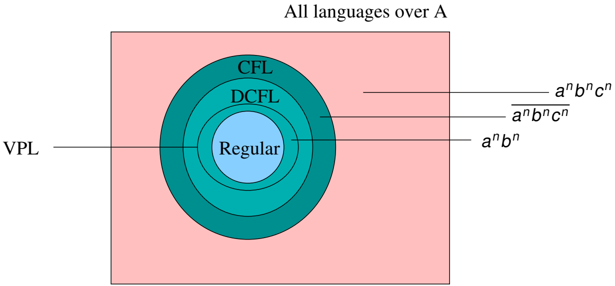
Restrictions:

- $\perp$ is never pushed on the stack
- Pop transitions can read $\perp$ from the stack but must leave it in place.
- No epsilon transitions

Run of M on a word $w = a_1 a_2 \ldots a_n$.

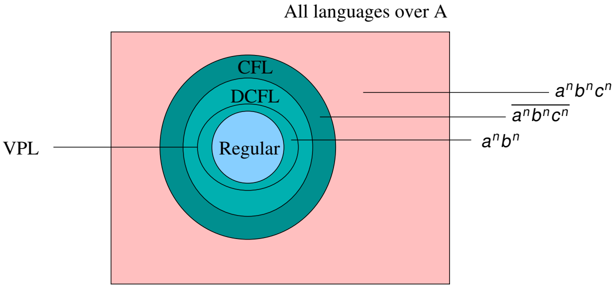
Class of languages accepted by VPA's are called Visibly Pushdown Languages (VPL).

Closure Properties of VPL



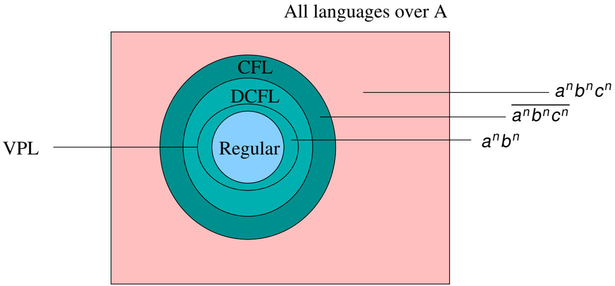
	Closed?
Union	

Closure Properties of VPL



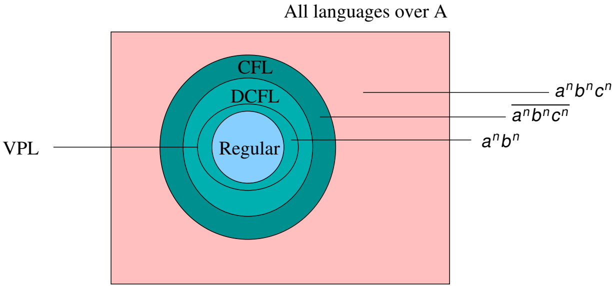
	Closed?
Union	√
Intersection	

Closure Properties of VPL



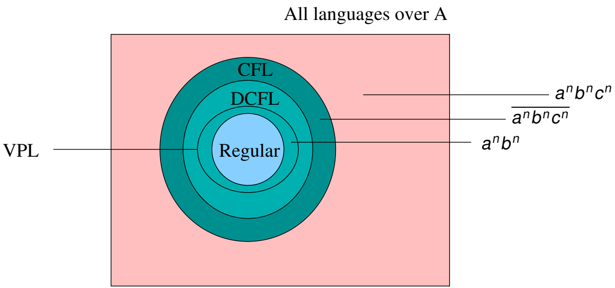
	Closed?
Union	√
Intersection	√
Concatentation	

Closure Properties of VPL



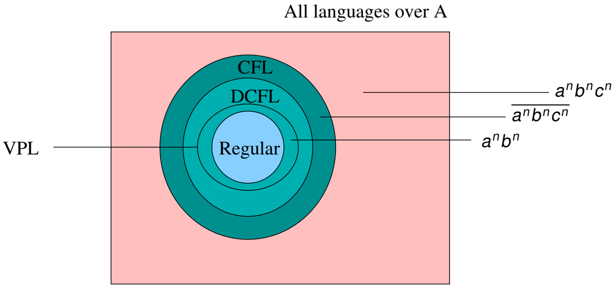
	Closed?
Union	√
Intersection	√
Concatentation	√
Kleene-*	

Closure Properties of VPL



	Closed?
Union	√
Intersection	√
Concatenation	√
Kleene-*	√
Complementation	

Closure Properties of VPL



	Closed?
Union	√
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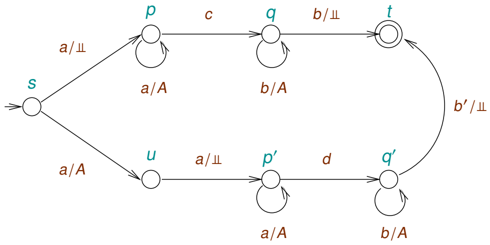
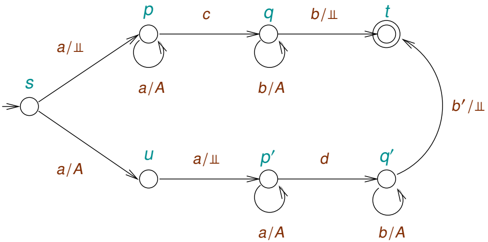
Complementation	√

Example non-deterministic VPA



Partitioned alphabet is $(\{a\}, \{b, b'\}, \{c, d\})$.

Example non-deterministic VPA

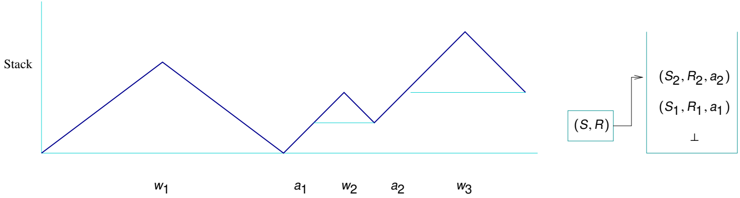


Partitioned alphabet is $(\{a\}, \{b, b'\}, \{c, d\})$.
Accepts language $\{a^n c b^n \mid n \geq 1\} \cup \{a^n d b^{n-2} b' \mid n \geq 1\}$.

Determinizing VPA's

Let $M = (Q, Q_0, \Gamma, \delta, F)$ be a VPA over $\widetilde{\Sigma}$. We define a new VPA M' as follows:

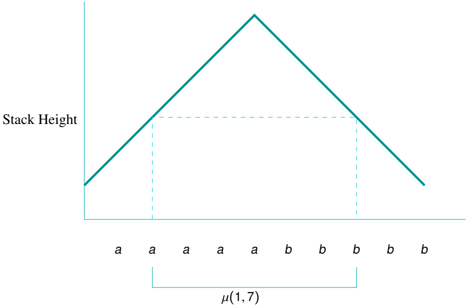
- Control state is of the form (S, R) where $S \subseteq Q \times Q$ and $R \subseteq Q$.
- Stack symbols will be of the form (S, R, a) where S and R are as above, and $a \in \Sigma_c$ is a call alphabet.
- Construction maintains the following invariant:
  - S_1 is the summary of w_1, S_2 of w_2, and S of w_3.
  - R_1 is the reach set after w_1, R_2 after $w_1 a_1 w_2$, and R after w.



Decision Procedures for VPLs

- Emptiness
- Language inclusion / equivalance
- Universality

MSO over $\widetilde{\Sigma}$ with matching predicate



- Interpreted over finite words w over Σ.
- Syntax:

$$Q_a(x) \mid x < y \mid \mu(x,y) \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \exists x \varphi \mid \exists X \varphi.$$

- $\mu(x,y)$ is true if $w(x)$ is a call and matching return is at $w(y)$.
- Example over $(\{a\},\{b\},\{d\})$:
  $\forall x(Q_a(x) \implies \exists y(\mu(x,y) \wedge Q_b(y)))$.

Logical characterization of VPLs

Theorem

L is a VPL over $\widetilde{\Sigma}$ iff L is definable in MSO($\widetilde{\Sigma}$).

Proof is similar to that of regular langauages.