Floyd-Hoare Style Program Verification

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Outline of these lectures

Floyd-Hoare Style of Program Verification

Robert W. Floyd: "Assigning meanings to programs" *Proceedings of the American Mathematical Society Symposia on Applied Mathematics* (1967)

C A R Hoare: "An axiomatic basis for computer programming", *Communications of the ACM* (1969).

Floyd-Hoare Logic

- A way of asserting properties of programs.
- Hoare triple: $\{A\}P\{B\}$ asserts that "Whenever program P is started in a state satisfying condition A, if it terminates, it will terminate in a state satisfying condition B."
- Example assertion: $\{n \geq 0\}\ P\ \{a = n + m\}$, where P is the program:

```
int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}
```

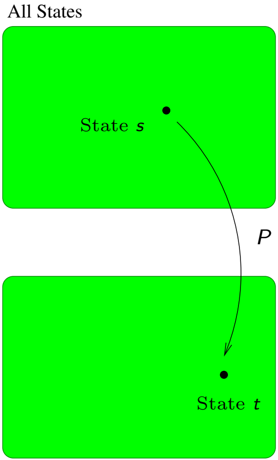
- Inductive Annotation ("consistent interpretation") (due to Floyd)
- A proof system (due to Hoare) for proving such assertions.
- A way of reasoning about such assertions using the notion of "Weakest Preconditions" (due to Dijkstra).

A simple programming language

- `skip`
- `x := ` e (assignment)
- `if` b `then` S `else` T (if-then-else)
- `while` b `do` S (while)
- S `;` T (sequencing)

Programs as State Transformers

View program P as a partial map $[P] : \mathit{Stores} \rightarrow \mathit{Stores}$. (Assume that $\mathit{Stores} = \mathit{Var} \rightarrow \mathbb{Z}$.)

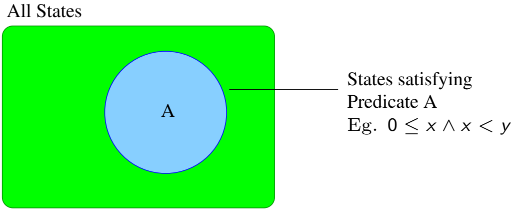


$\langle x \mapsto 2,\ y \mapsto 10,\ z \mapsto 3 \rangle$

```
y := y + 1;
z := x + y
```

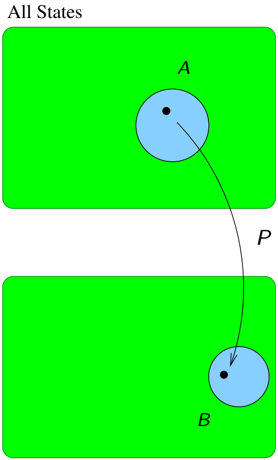
$\langle x \mapsto 2,\ y \mapsto 11,\ z \mapsto 13 \rangle$

Predicates on States



Assertion of "Partial Correctness" $\{A\}P\{B\}$

$\{A\}P\{B\}$ asserts that "Whenever program P is started in a state satisfying condition A, either it will not terminate, or it will terminate in a state satisfying condition B."



$\{10 \leq y\}$

```
y := y + 1;
z := x + y
```

$\{x < z\}$

Mathematical meaning of a Hoare triple

- View program P as a relation

$$[P] \subseteq \mathrm{Stores} \times \mathrm{Stores}.$$

  so that $(s, t) \in [P]$ iff it is possible to start P in the state s and terminate in state t.
- As usual here elements of Stores are maps from variables to integers.
- $[P]$ is possibly non-determinisitic, in case we also want to model non-deterministic assignment etc.
- Then the Hoare triple $\{A\}\ P\ \{B\}$ is true iff for all states s and t: whenever $s \models A$ and $(s, t) \in [P]$, then $t \models B$.
- In other words $Post_{[P]}([A]) \subseteq [B]$.

Example programs and pre/post conditions

```
// Pre: true

if (a <= b)
  min := a;
else
  min := b;

// Post: min <= a && min <= b
```

```
// Pre: 0 <= n

int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}

// Post: a = m + n
```

Hoare's view: Program as a composition of statements

```
int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}
```

Hoare's view: Program as a composition of statements

```
int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}
```

```
S1: int a := m;
S2: int x := 0;
S3: while (x < n) {
      a := a + 1;
      x := x + 1;
    }
```

Program is `S1;S2;S3`

Proof rules of Hoare Logic

Axiom of Valid formulas:

$$\frac{}{A}$$

provided "$\models A$" (i.e. A is a valid logical formula, eg. $x > 10 \implies x > 0$).

Skip:

$$\frac{}{\{A\}\ \texttt{skip}\ \{A\}}$$

Assignment

$$\frac{}{\{A[e/x]\}\ \texttt{x := e}\ \{A\}}$$

Proof rules of Hoare Logic

If-then-else:

$$\frac{\{P \wedge b\}\ S\ \{Q\},\ \{P \wedge \neg b\}\ T\ \{Q\}}{\{P\}\ \texttt{if}\ \ b\ \ \texttt{then}\ \ S\ \ \texttt{else}\ \ T\ \{Q\}}$$

While (here P is called a *loop invariant*)

$$\frac{\{P \wedge b\}\ S\ \{P\}}{\{P\}\ \texttt{while}\ \ b\ \ \texttt{do}\ \ S\ \{P \wedge \neg b\}}$$

Sequencing:

$$\frac{\{P\}\ S\ \{Q\},\ \{Q\}\ T\ \{R\}}{\{P\}\ S\,;T\ \{R\}}$$

Weakening:

$$\frac{P \implies Q,\ \{Q\}\ S\ \{R\},\ R \implies T}{\{P\}\ S\ \{T\}}$$

Loop invariants

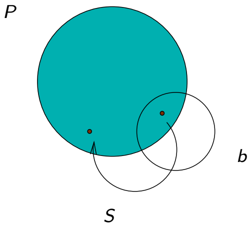
A predicate P is a loop invariant for the while loop:

```
while (b) {
  S
}
```

if $\{P \wedge b\}\ S\ \{P\}$ holds.

If P is a loop invariant then we can infer that:

$$\{P\}\ \texttt{while}\ b\ \texttt{do}\ S\ \{P \wedge \neg b\}$$



Some examples to work on

Use the rules of Hoare logic to prove the following assertions:

1. $\{x \geq 3\}$ `x := x + 2` $\{x \geq 5\}$
2. $\{(y \leq 0) \wedge (x > -1)\}$ `if` $(y < 0)$ `then x:=x+1 else x:=y` $\{x > 0\}$
3. $\{x \leq 0\}$ while $(x \leq 5)$ do x := x+1 $\{x = 6\}$

Exercise

Prove using Hoare logic:

$$\{n \geq 1\}\ P\ \{a = n!\},$$

where P is the program:

```
x := n;
a := 1;
while (x ≥ 1) {
    a := a * x;
    x := x - 1
}
```

Assume that factorial is defined as follows:

$$n! = \begin{cases} n \times (n-1) \times \cdots \times 1 & \text{if} \quad n \geq 1 \\ 1 & \text{if} \quad n = 0 \\ -1 & \text{if} \quad n < 0 \end{cases}$$

Exercise

Prove using Hoare logic:

$$\{n \geq 1\}\ P\ \{a = n!\},$$

where P is the program:

```
S1:  x := n;
S2:  a := 1;
S3:  while (x ≥ 1) {
S4:      a := a * x;
S5:      x := x - 1
     }
```

Assume that factorial is defined as follows:

$$n! = \begin{cases} n \times (n-1) \times \cdots \times 1 & \text{if} \quad n \geq 1 \\ 1 & \text{if} \quad n = 0 \\ -1 & \text{if} \quad n < 0 \end{cases}$$

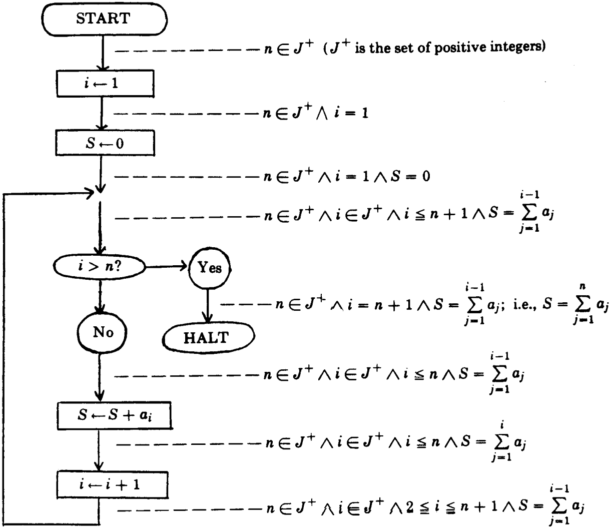
Solution

Need a loop invariant P satisfying:

1. $\{n \geq 1\}\ S1; S2\ \{P\}$
2. $\{P \wedge (x \geq 1)\}\ S4; S5\ \{P\}$
3. $(P \wedge \neg(x \geq 1)) \implies (a = n!)$

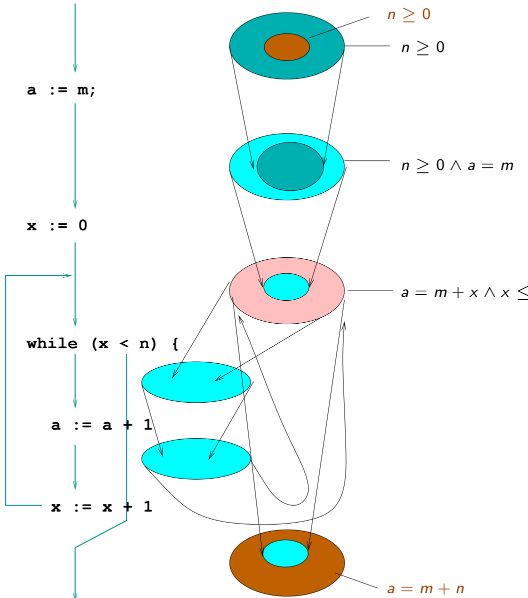
A potential P: $(x \geq 0) \wedge (a \times x! = n!)$.

Floyd's style of proof: Inductive Annotation



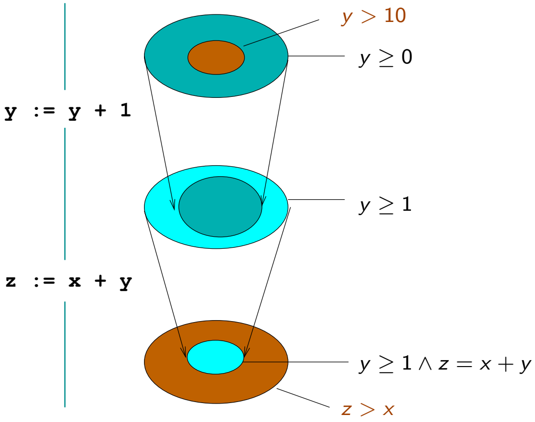
Inductive annotation based proof of a pre/post specification

- Annotate each program point i with a predicate A_i
- Successive annotations must be inductive: $A_i \wedge [S_i] \implies A'_{i+1}$.
- Annotation is adequate: $Pre \implies A_1$ and $A_n \implies Post$.
- Adequate annotation constitutes a proof of $\{Pre\}$ Prog $\{Post\}$.



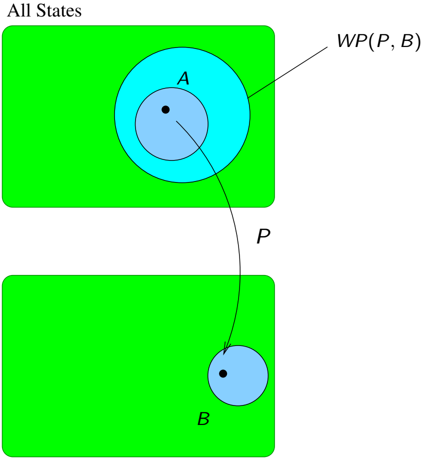
Example of inductive annotation

To prove: $\{y > 10\}$ y := y+1; z := x+y $\{z > x\}$



Weakest Precondition $WP(P, B)$

$WP(P, B)$ is "a predicate that describes the exact set of states s such that when program P is started in s, if it terminates it will terminate in a state satisfying condition B."



$\{10 < y\}$

```
y := y + 1;
z := x + y;
```

$\{x < z\}$

Exercise: Give "weakest" preconditions

1. $\{? \qquad\}$ `x := x + 2` $\{x \geq 5\}$

Exercise: Give "weakest" preconditions

1. $\{\ x \geq 3\}$ `x := x + 2` $\{x \geq 5\}$

2. $\{? \qquad\qquad\qquad\qquad\qquad\qquad\}$
   `if (y < 0) then x := x+1 else x := y`
   $\{x > 0\}$

Exercise: Give "weakest" preconditions

1. $\{\ x \geq 3\}$ `x := x + 2` $\{x \geq 5\}$

2. $\{\ (y < 0 \wedge x > -1) \vee (y > 0)\}$
   `if (y < 0) then x := x+1 else x := y`
   $\{x > 0\}$

3. $\{? \qquad \}$ while $(x \leq 5)$ do x := x+1 $\{x = 6\}$

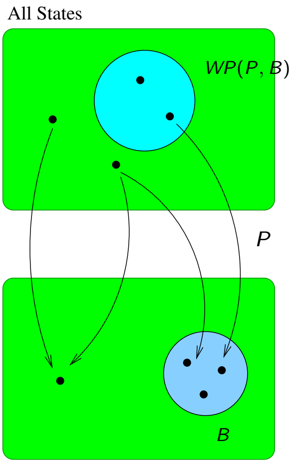
Exercise: Give "weakest" preconditions

1. $\{\ x \geq 3\}$ `x := x + 2` $\{x \geq 5\}$

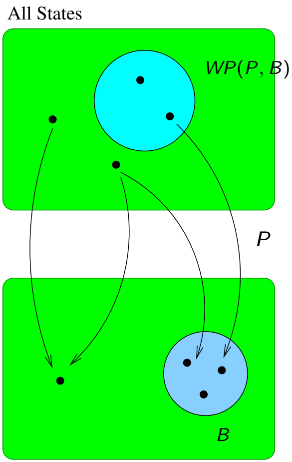
2. $\{\ (y < 0 \wedge x > -1) \vee (y > 0)\}$
   `if (y < 0) then x := x+1 else x := y`
   $\{x > 0\}$

3. $\{\ x \leq 6\}$ while $(x \leq 5)$ do x := x+1 $\{x = 6\}$

Exercise: How will you define $WP(P, B)$?



Exercise: How will you define $WP(P, B)$?



$$WP(P, B) = \{s \mid \forall t : (s, t) \in [P] \text{ we have } t \models B\}$$

Rules for Computing Weakest Precondition

For assignment statement `x = e`:

$\{B[e/x]\}$

```
x = e;
```

$\{B\}$

Rules for Computing Weakest Precondition

For assignment statement `x = e`:

$\{B[e/x]\}$

```
x = e;
```

$\{B\}$

$\{(x + y) > 0 \wedge y = 0\}$

```
z = x + y;
```

$\{z > 0 \wedge y = 0\}$

Rules for Computing Weakest Precondition

If-then-else statement `if c then` S_1 `else` S_2:

$$\{(c \wedge WP(S_1, B)) \vee (\neg c \wedge WP(S_2, B))\}$$

```
if (c)
   S1;
else
   S2;
```

$$\{B\}$$

Rules for Computing Weakest Precondition

If-then-else statement `if c then` S_1 `else` S_2:

$\{(c \wedge WP(S_1, B)) \vee (\neg c \wedge WP(S_2, B))\}$

```
if (c)
   S1;
else
   S2;
```

$\{B\}$

$\{((x < y) \wedge (y > w)) \vee ((x \geq y) \wedge (x > w))\}$

```
if (x < y)
   z = y;
else
   z = x;
```

$\{z > w\}$

WP rule for sequencing

$$WP(S;T,\ B) = WP(S, WP(T, B)).$$

Weakest Precondition for `while` statements

- We can "approximate" $WP(\texttt{while}\ b\ \texttt{do}\ c)$.
- $WP_i(w, A) =$ the set of states from which the body c of the loop is either entered more than i times or we exit the loop in a state satisfying A.
- WP_i defined inductively as follows:

$$
\begin{array}{lcl}
WP_0 & = & b \vee A \\
WP_{i+1} & = & (\neg b \wedge A) \vee (b \wedge WP(c, WP_i))
\end{array}
$$

- Then $WP(w, A)$ can be shown to be the "limit" or least upper bound of the chain $WP_0(w, A)$, $WP_1(w, A), \ldots$ in a suitably defined lattice (here the join operation is "And" or intersection).

Illustration of WP_i through example

Consider the program w below:

```
while (x ≥ 10) do
    x := x - 1
```

- What is the weakest precondition of w with respect to the postcondition $(x \leq 0)$?
- Compute $WP_0(w, (x \leq 0))$, $WP_1(w, (x \leq 0))$, $\ldots$.

Illustration of WP_i through example

Consider the program w below:

```
while (x ≥ 10) do
    x := x - 1
```

- What is the weakest precondition of w with respect to the postcondition $(x \leq 0)$?
- Compute $WP_0(w, (x \leq 0))$, $WP_1(w, (x \leq 0))$, ...

Using weakest preconditions in inductive proofs

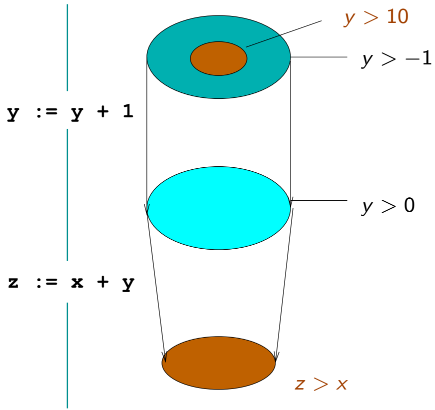
Weakest preconditions give us a way to:

- Check inductiveness of annotations

$$\{A_i\}\ S_i\ \{A_{i+1}\} \text{ iff } A_i \implies WP(S_i, A_{i+1})$$

- Reduce the amount of user-annotation needed
  - Programs without loops don't need any user-annotation
  - For programs with loops, user only needs to provide loop invariants

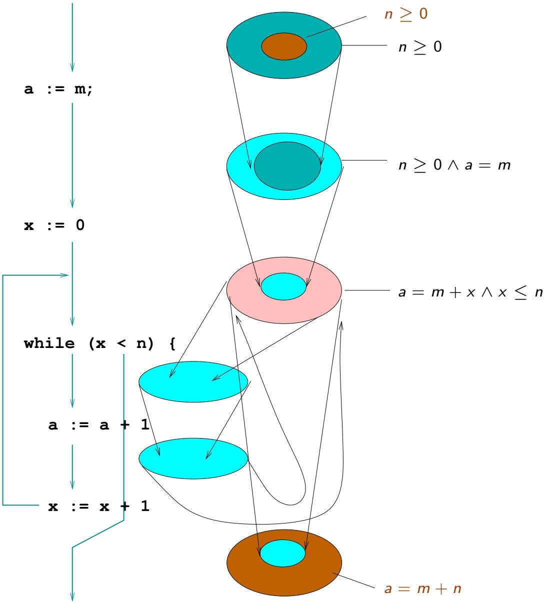
Checking $\{A\}$ P $\{B\}$ using WP



Check that

$$(y > 10) \implies WP(P, z > x)$$

Example proof of add program



Reducing verification to satisfiability: Generating Verification Conditions

To check:

$\{y > 10\}$

```
y := y + 1;
z := x + y;
```

$\{x < z\}$

Use the weakest precondition rules to generate the verification condition:

$$(y > 10) \implies (y > -1).$$

Check the verification condition by asking a theorem prover / SMT solver if the formula

$$(y > 10) \wedge \neg(y > -1).$$

is satisfiable.

What about `while` loops?

```
Pre: 0 <= n

int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}

Post: a = m + n
```

Adequate loop invariant

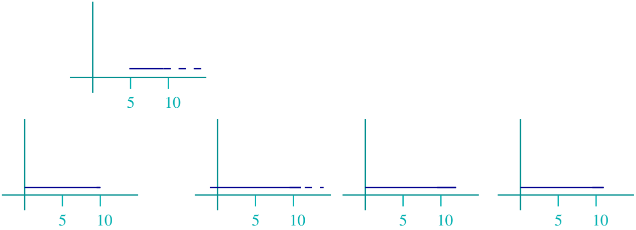
What is a “good” loop invariant for this program?

```
x := 0;
while (x < 10) {
 if (x >= 0)
  x := x + 1;
 else
  x := x - 1;
}
assert(x <= 11);
```

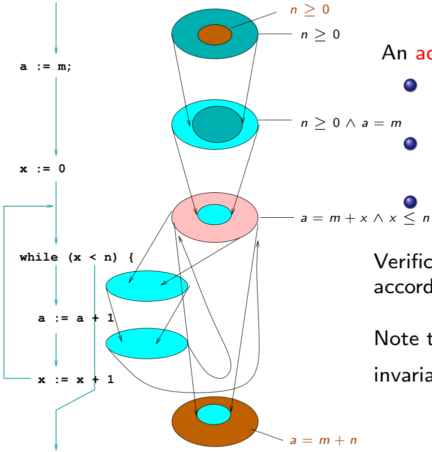
Adequate loop invariant

```
x := 0;
while (x < 10) {
 if (x >= 0)
  x := x + 1;
 else
  x := x - 1;
}
assert(x <= 11);
```

Canonical Invariant	*Not−inv*	*Inv,not−ind*	*Inv,ind,not−adeq*	*Inv,ind,adeq*
$0 \leq x \leq 10$	$5 \leq x$	$-1 \leq x$	$0 \leq x \leq 12$	$0 \leq x \leq 11$



Adequate loop invariant



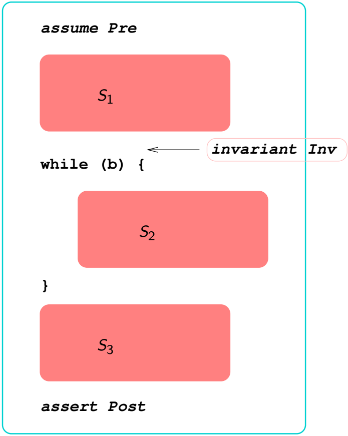
An adequate loop invariant needs to satisfy:

- $\{n \geq 0\}$ `a := m; x := 0` $\{a = m + x \wedge x \leq n\}$.
- $\{a = m + x \wedge x \leq n \wedge x < n\}$ `a := a+1; x := x+1` $\{a = m + x \wedge x \leq n\}$.
- $\{a = m + x \wedge x \leq n \wedge x \geq n\}$ `skip` $\{a = m + n\}$.

Verification conditions are generated accordingly.

Note that $a = m + x$ is not an adequate loop invariant.

Generating Verification Conditions for a program



The following VCs are generated:

- $Pre \wedge [S_1] \implies Inv'$
  Or: $Pre \implies WP(S_1, Inv)$
- $Inv \wedge b \wedge [S_2] \implies Inv'$
  Or: $(Inv \wedge b) \implies WP(S_2, Inv)$
- $Inv \wedge \neg b \wedge [S_3] \implies Post'$
  Or: $Inv \wedge \neg b \implies WP(S_3, Post)$

Soundness and Completeness of Hoare logic

- Hoare logic is sound (i.e. if we can prove "$\{A\}\ P\ \{B\}$" in the logic, then $\{A\}\ P\ \{B\}$ is true.)
  - Prove that each axiom and each rule is sound
- Conversely, is it complete? That is, if $\{A\}\ P\ \{B\}$ is true for a program P and pre/post-conditions A and B, does there exists a proof tree for $\{A\}\ P\ \{B\}$ using the rules of Hoare logic?
- Yes, provided the assertion logic L can express all "weakest preconditions" (for all programs, and post-conditions expressed in L).

Relative completeness of Hoare logic

Theorem (Cook 1974)

Hoare logic is complete provided the assertion language L can express the WP for any program P and post-condition B.

Proof uses WP predicates and proceeds by induction on the structure of the program P.

- Suppose $\{A\}$ `skip` $\{B\}$ holds. Then it must be the case that $A \implies B$ is true. By Skip rule we know that $\{B\}$ `skip` $\{B\}$. Hence by Weakening rule, we get that $\{A\}$ `skip` $\{B\}$ holds.
- Suppose $\{A\}$ `x := e` $\{B\}$ holds. Then it must be the case that $A \implies B[e/x]$. By Assignment rule we know that $\{B[e/x]\}$ `x := e` $\{B\}$ is true. Hence by Weakening rule, we get that $\{A\}$ `x := e` $\{B\}$ holds.
- Similarly for `if-then-else`.

Relative completeness of Hoare logic

- Suppose $\{A\}$ `while b do S` $\{B\}$ holds. Let $P = WP(\texttt{while b do S}, B)$. Then it is not difficult to check that P is a loop invariant for the while statement. I.e $\{P \wedge b\}$ `S` $\{P\}$ is true. By induction hypothesis, this triple must be provable in Hoare logic. Hence we can conclude using the While rule, that $\{P\}$ `while b do S` $\{P \wedge \neg b\}$. But since P was a valid precondition, it follows that $(P \wedge \neg b) \implies B$. By the weakening rule, we have a proof of $\{A\}$ `while b do S` $\{B\}$.

Conclusion

- Hoare's style of proving programs views the program as a sequential composition of programs and constructs a proof tree.
- Floyd's style views the control-flow graph of the program, with annotations at each program point.
- Proofs in one style can be translated to the other.
- Using weakest preconditions we can generate verification conditions, to reduce verification to checking validity of a logical formula.
- Can be extended to handle functions (using function contracts), arrays (quantification), concurrency (Rely-Guarantee/Owicki-Gries styles).

Main challenge is the need for user annotation (adequate loop invariants, function contracts).
Can be increasingly automated (using learning techniques).