Floyd-Hoare Style Program Verification (FMSE Course)

Deepak D'Souza

Department of Computer Science and Automation
Indian Institute of Science, Bangalore.

27 Feb 2024

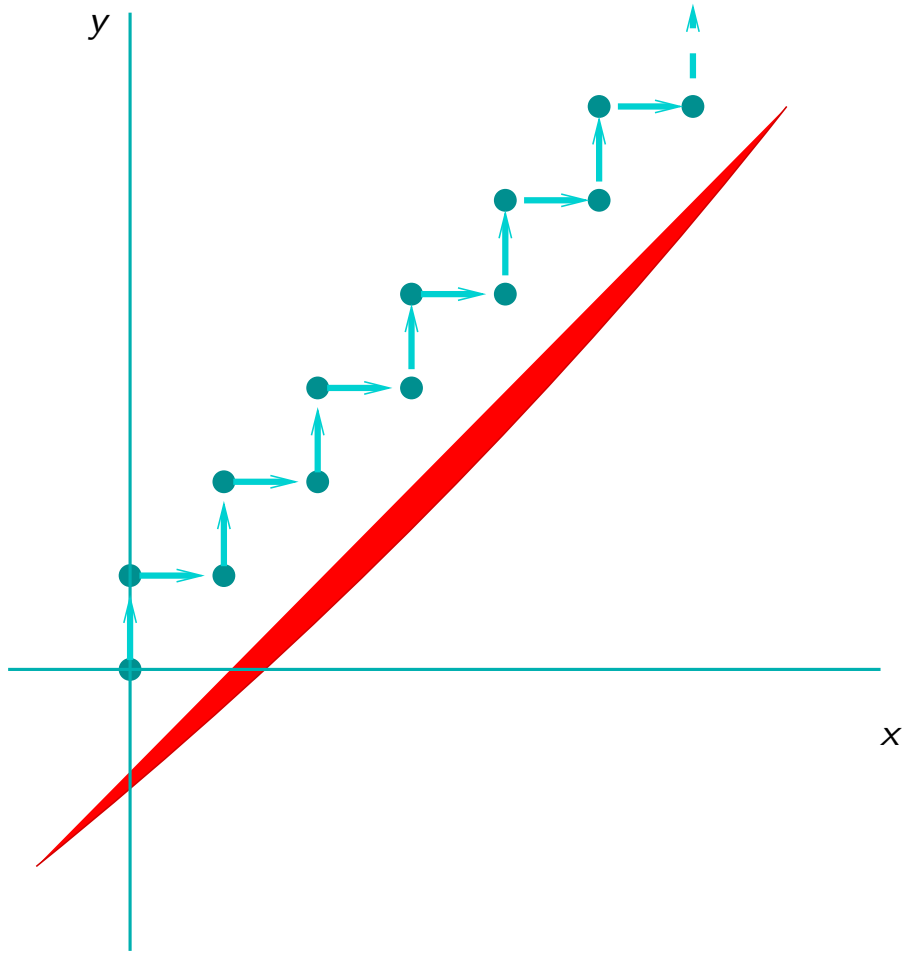
Outline of these lectures

Checking Pre/Post Assertions in Programs

- Moving on from reasoning about models to reasoning about code.
- Still a deductive style of verification.
- Helps us to verify assertions and also refinement-based functionality verification.

Example Program and Property

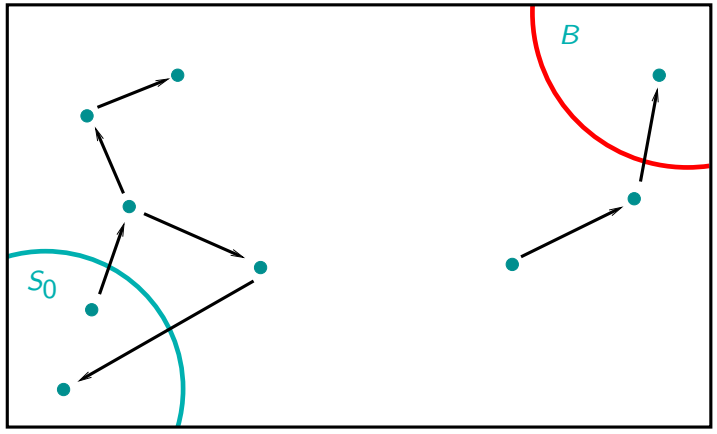
```
x := 0;
y := 0;
while (*) {
  if (x < y)
    x++;
  else
    y++;
}
// assert y != x - 1
```



How would one check that this program satisfies the given assertion?

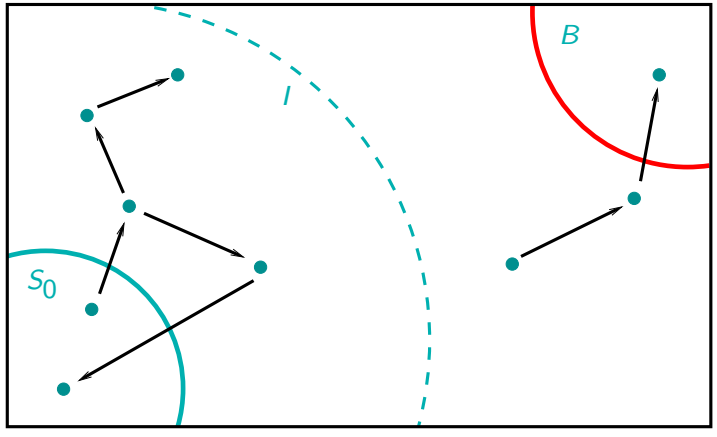
Idea of Deductive Verification

Problem: Given a transition system $\mathcal{T} = (S, S_0, \rightarrow)$ and an set of unsafe states $B \subseteq S$, does an execution of $\mathcal{T}$ reach a state in B?



Find a set of states I such that

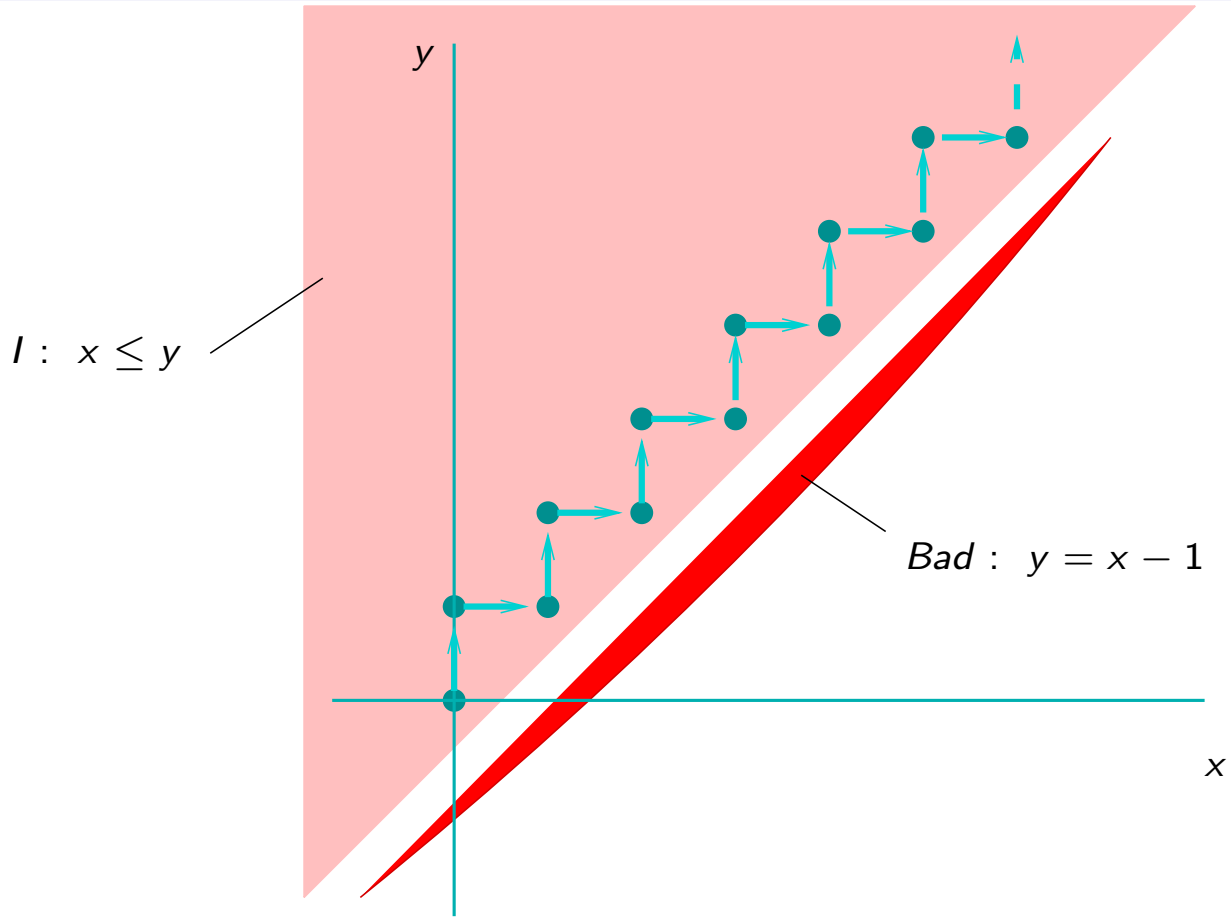
1. $S_0 \subseteq I$ (initial states belong to I)
2. $s \in I$ and $s \rightarrow s'$, implies $s' \in I$ (I is inductive wrt trans)
3. $I \cap B = \emptyset$ (I disjoint from Bad states).



Such an I is called an adequate inductive invariant.

Idea of deductive verification

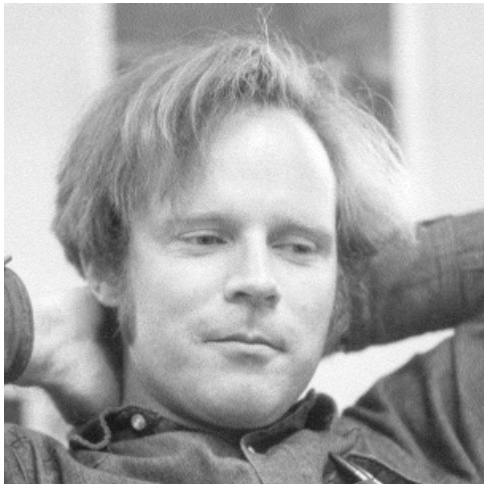
```
x := 0;
y := 0;
while (*) {
  if (x < y)
    x++;
  else
    y++;
}
// assert y != x - 1
```



I is an adequate inductive invariant:

1. $s_0 \in I$ (initial state belongs to I)
2. $s \in I$ and $s \rightarrow s'$, implies $s' \in I$ (I is inductive wrt trans)
3. $I \cap B = \emptyset$ (I disjoint from Bad states).

Floyd-Hoare Style of Program Verification



Robert W. Floyd: "Assigning meanings to programs" *Proceedings of the American Mathematical Society Symposia on Applied Mathematics* (1967)

C A R Hoare: "An axiomatic basis for computer programming", *Communications of the ACM* (1969).

Floyd-Hoare Logic

- A way of asserting properties of programs.
- Hoare triple: $\{A\}P\{B\}$ asserts that "Whenever program P is started in a state satisfying condition A, if it terminates, it will terminate in a state satisfying condition B."
- Example assertion: $\{n \geq 0\}\ P\ \{a = n + m\}$, where P is the program:

```
int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}
```

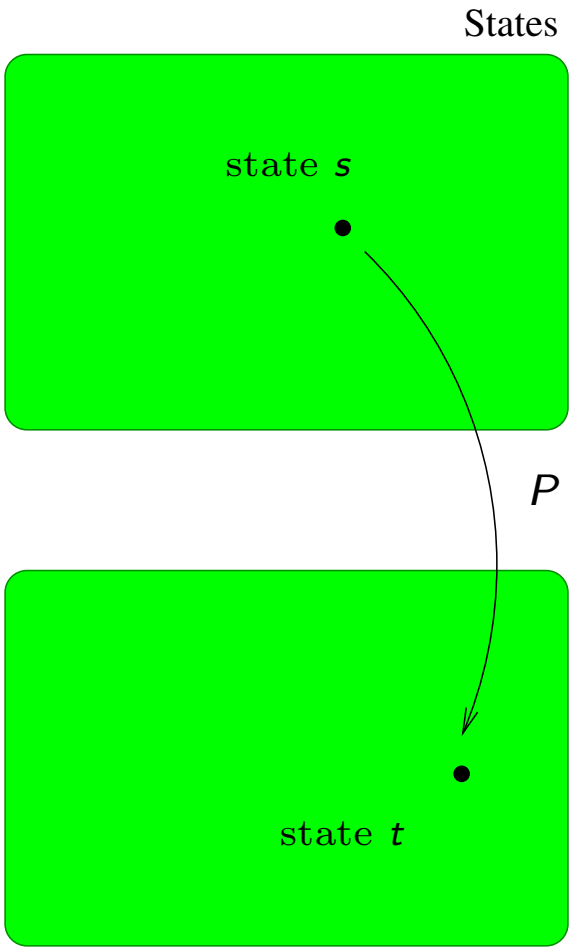
- Inductive Annotation ("consistent interpretation") (due to Floyd)
- A proof system (due to Hoare) for proving such assertions.
- A way of reasoning about such assertions using the notion of "Weakest Preconditions" (due to Dijkstra).

A Simple Programming Language

- `skip` (do nothing)
- `x := e` (assignment)
- `if` b `then` S `else` T (if-then-else)
- `while` b `do` S (while loop)
- S ; T (sequencing)

Programs as State Transformers

- Program state is a valuation to variables of the program: $States = Var \to \mathbb{Z}$.
- View program P as a partial map $[\![P]\!] : States \to States$.

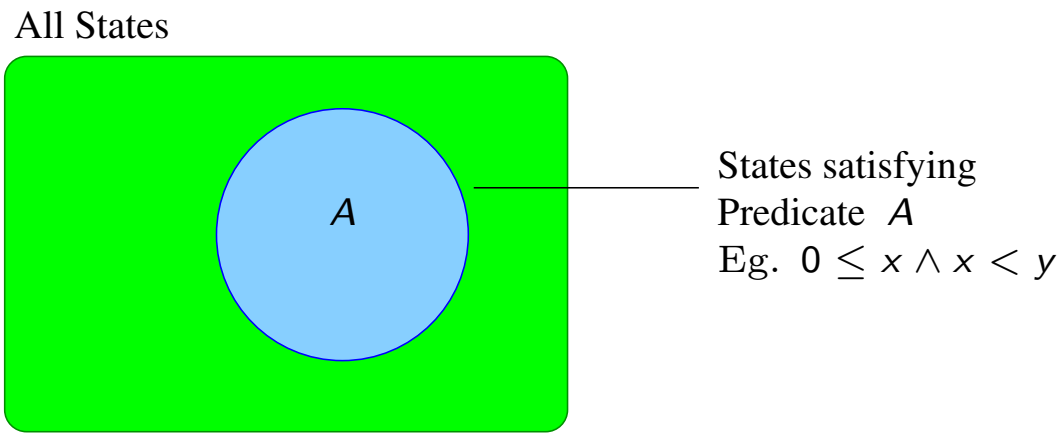


$s : \langle x \mapsto 2,\ y \mapsto 10,\ z \mapsto 3 \rangle$

```
y := y + 1;
z := x + y
```

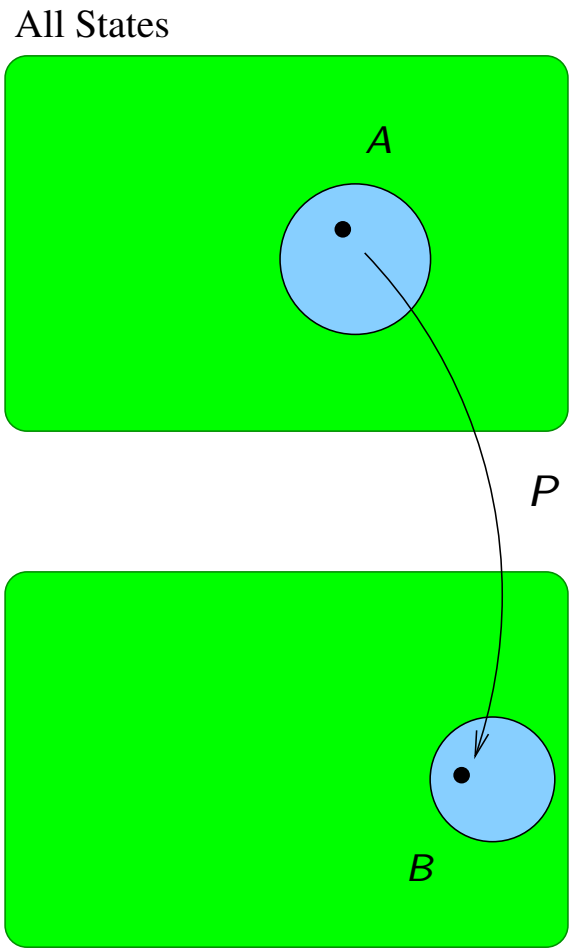
$t : \langle x \mapsto 2,\ y \mapsto 11,\ z \mapsto 13 \rangle$

Predicates on States



Assertion of "Partial Correctness" $\{A\}P\{B\}$

$\{A\}P\{B\}$ asserts that "Whenever program P is started in a state satisfying condition A, either it will not terminate, or it will terminate in a state satisfying condition B."



$\{10 \leq y\}$

```
y := y + 1;
z := x + y
```

$\{x < z\}$

Mathematical meaning of a Hoare triple

- View program P as a relation on States (allows non-termination as well as non-determinism)

$$[\![P]\!] \subseteq \text{States} \times \text{States}.$$

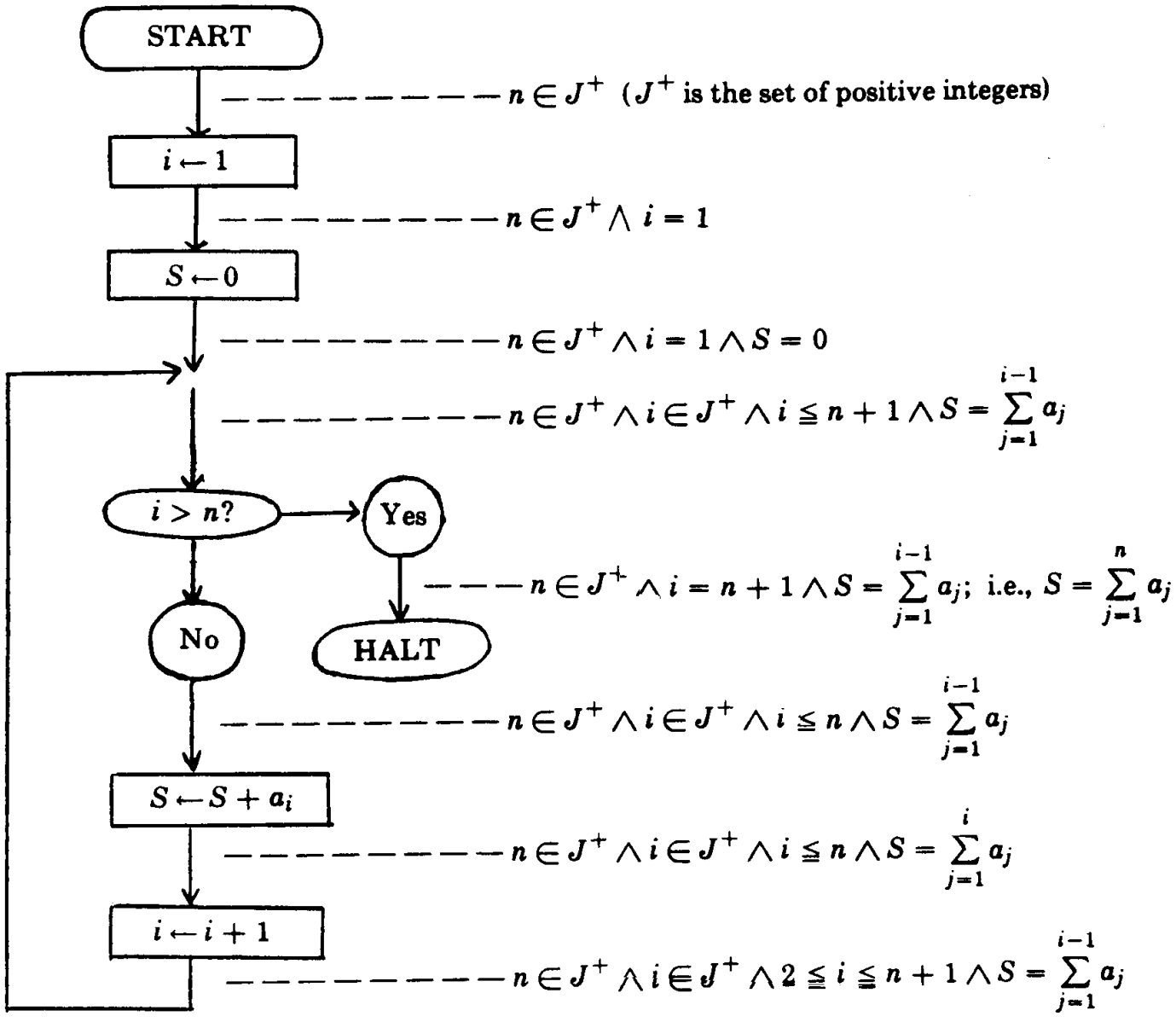
  Here $(s, t) \in [\![P]\!]$ iff it is possible to start P in the state s and terminate in state t.
- $[\![P]\!]$ is possibly non-determinisitic, in case we also want to model non-deterministic assignment etc.
- Then the Hoare triple $\{A\}\ P\ \{B\}$ is true iff for all states s and t: whenever $s \models A$ and $(s, t) \in [\![P]\!]$, then $t \models B$.
- In other words $Post_{[\![P]\!]}([\![A]\!]) \subseteq [\![B]\!]$.

Example programs and pre/post conditions

```
// Pre: true

if (a <= b)
  min := a;
else
  min := b;

// Post: min <= a && min <= b
```

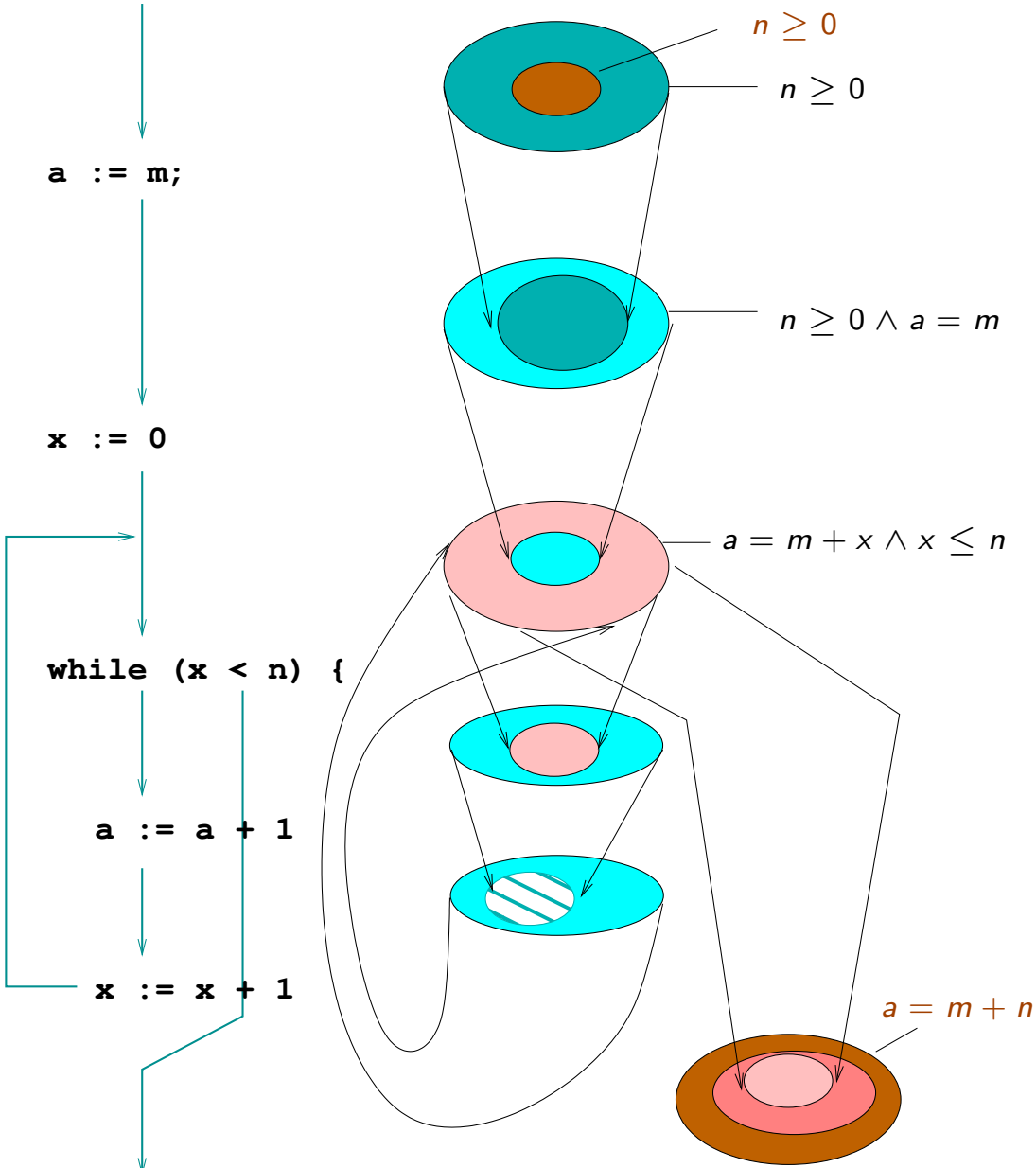
```
// Pre: 0 <= n

int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}

// Post: a = m + n
```

Floyd style proof: Inductive Annotation



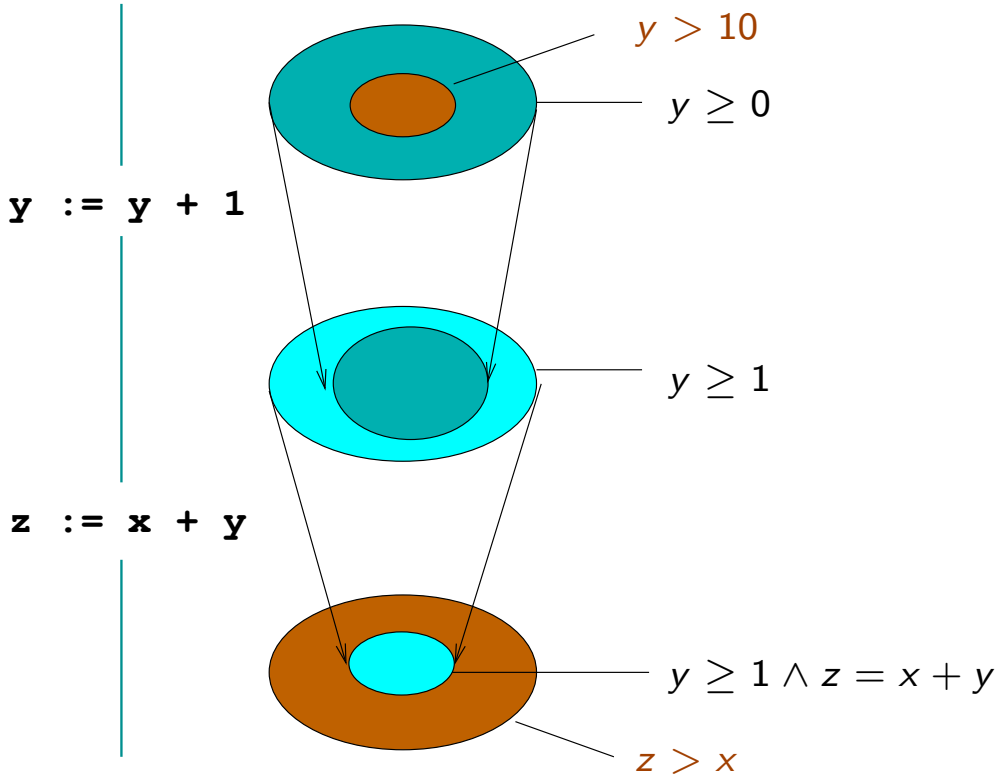
Inductive annotation based proof of a pre/post specification

- Annotate each program point i with a predicate A_i
- Successive annotations must be inductive: $[\![S_i]\!]([\![A_i]\!]) \subseteq [\![A_{i+1}]\!]$, OR logically: $A_i \wedge [S_i] \implies A'_{i+1}$.
- Annotation is adequate: $Pre \implies A_1$ and $A_n \implies Post$.
- Adequate annotation constitutes a proof of $\{Pre\}$ Prog $\{Post\}$.



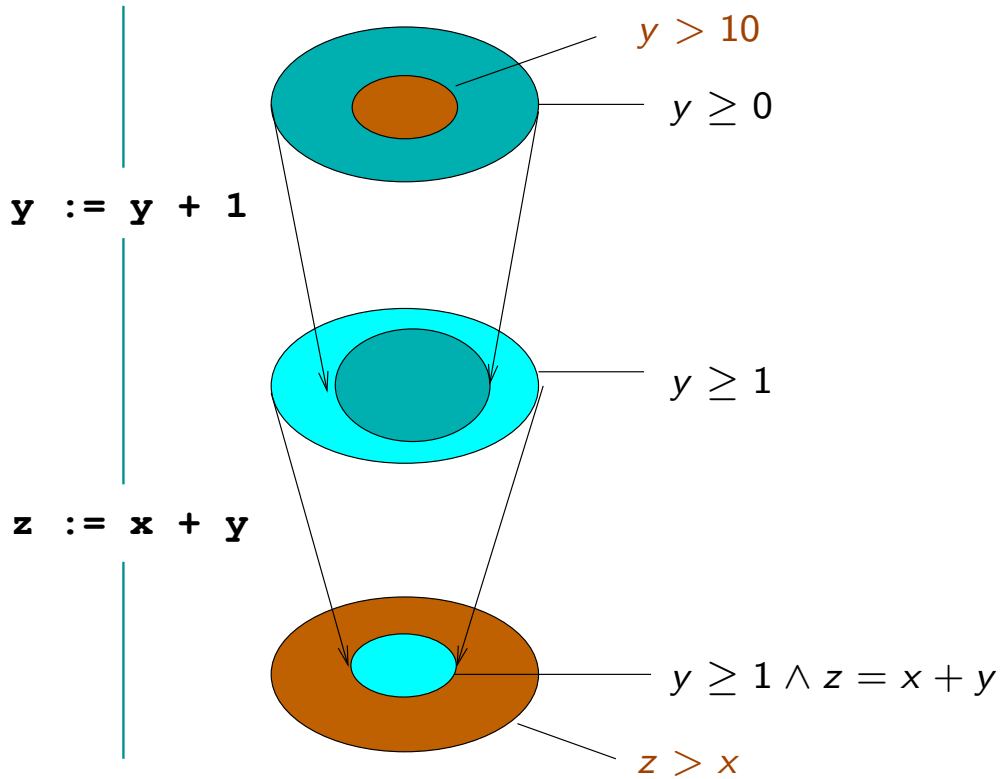
Example of inductive annotation

To prove: $\{y > 10\}$ `y := y+1; z := x+y` $\{z > x\}$



Example of inductive annotation

To prove: $\{y > 10\}$ `y := y+1; z := x+y` $\{z > x\}$



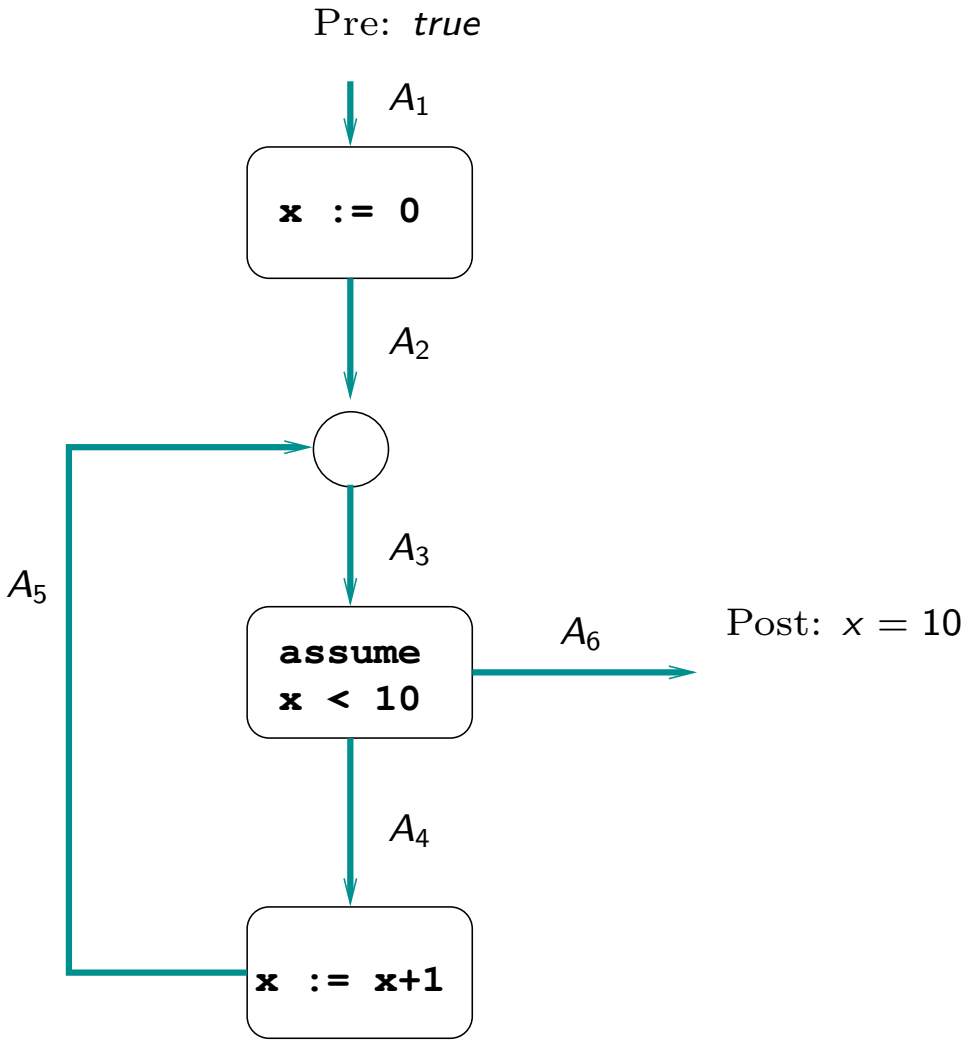
Logical proof obligations (VCs):

$$(y > 10 \implies y \geq 0) \wedge ((y \geq 1 \wedge z = x + y) \implies z > x) \wedge$$
$$((y \geq 0 \wedge y' = y + 1 \wedge x' = x \wedge z' = z) \implies y' \geq 1) \wedge$$
$$((y \geq 1 \wedge z' = x + y \wedge x' = x \wedge y' = y) \implies y' \geq 1 \wedge z' = x' + y')$$

Exercise

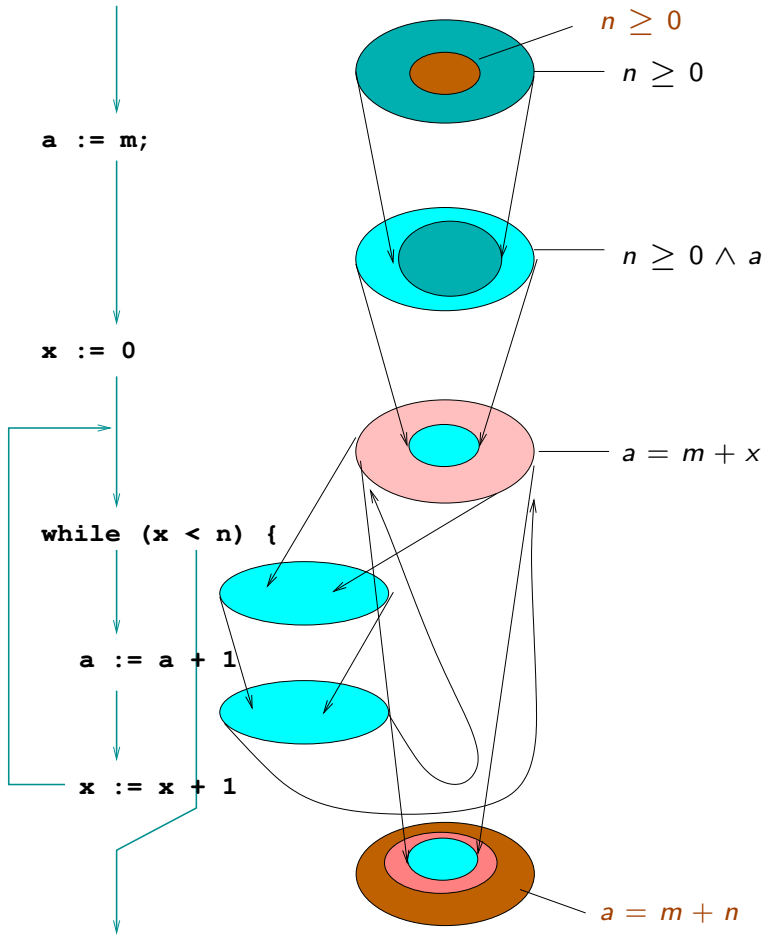
Prove using Floyd-style annotation:

```
// Pre: true
int x := 0;
while (x < 10)
  x := x + 1;
// Post: x = 10
```



Also write out the proof obligations (verification conditions).

Adequate loop invariant



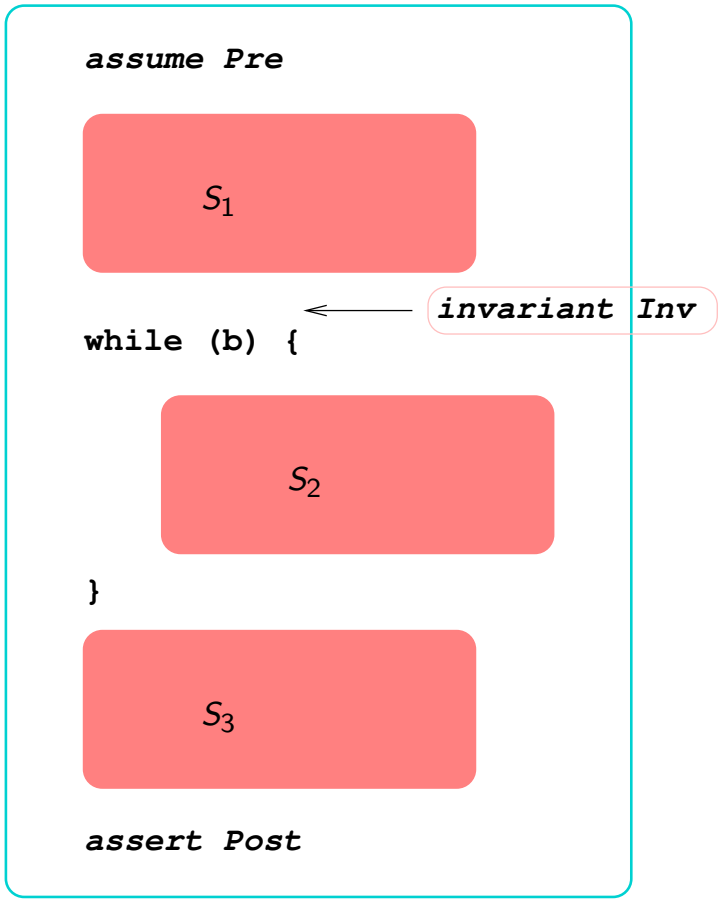
An adequate loop invariant needs to satisfy:

- $\{n \geq 0\}$ `a := m; x := 0` $\{a = m + x \wedge x \leq n\}$.
- $\{a = m + x \wedge x \leq n \wedge x < n\}$ `a := a+1; x := x+1` $\{a = m + x \wedge x \leq n\}$.
- $\{a = m + x \wedge x \leq n \wedge x \geq n\}$ `skip` $\{a = m + n\}$.

Verification conditions are generated accordingly.

Note that $a = m + x$ is not an adequate loop invariant.

Generating Verification Conditions for a program



The following VCs are generated:

- $Pre \wedge [S_1] \implies Inv'$
  Or: $Pre \implies WP(S_1, Inv)$
- $Inv \wedge b \wedge [S_2] \implies Inv'$
  Or: $(Inv \wedge b) \implies WP(S_2, Inv)$
- $Inv \wedge \neg b \wedge [S_3] \implies Post'$
  Or: $Inv \wedge \neg b \implies WP(S_3, Post)$

Adequate loop invariant

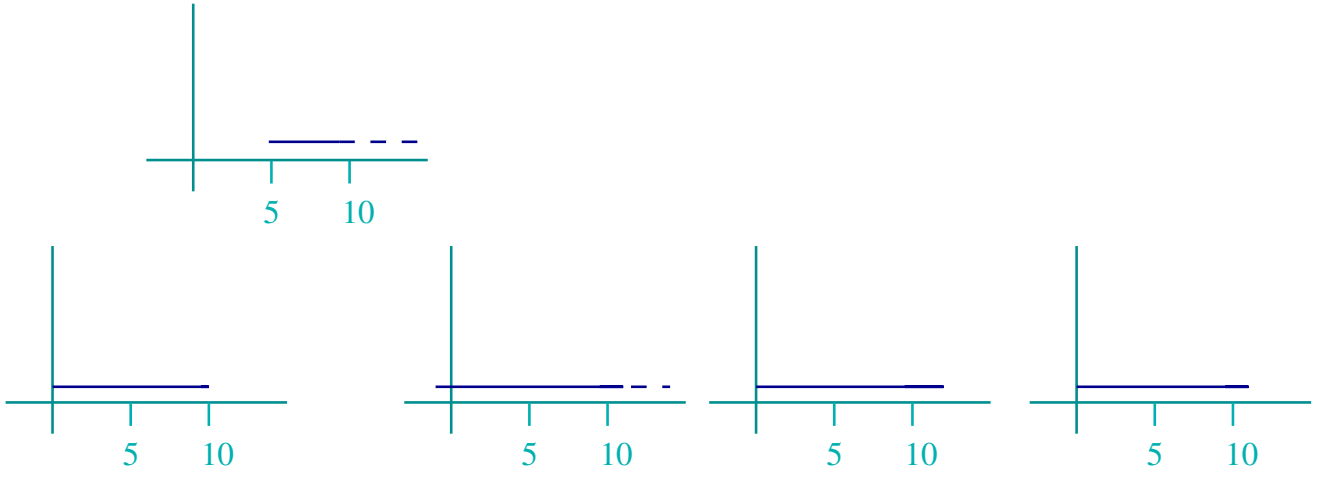
What is a "good" loop invariant for this program?

```
x := 0;
while (x < 10) {
 if (x >= 0)
  x := x + 1;
 else
  x := x - 1;
}
assert(x <= 11);
```

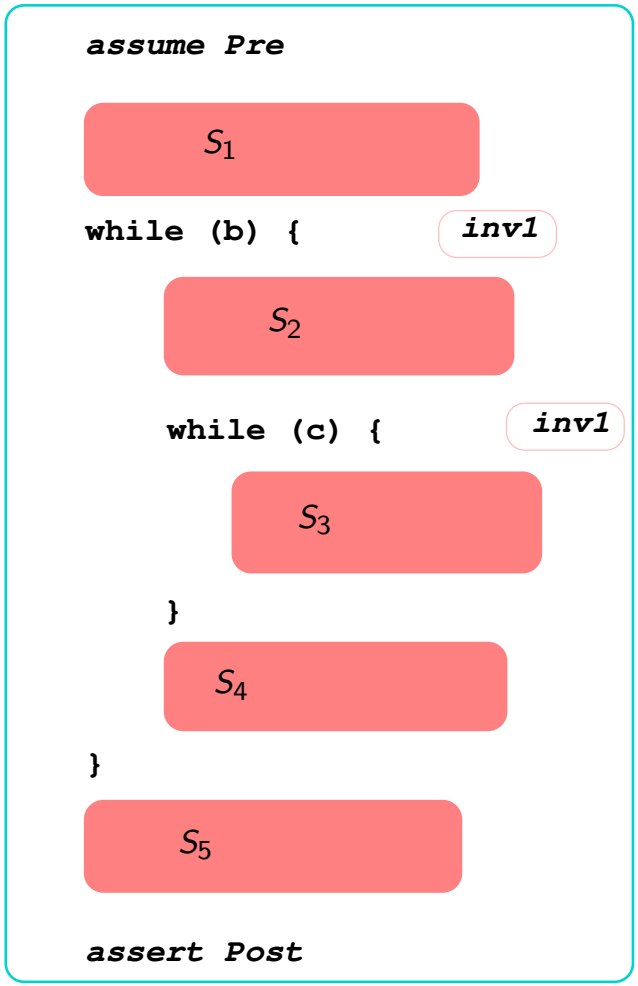
Adequate loop invariant

```
x := 0;
while (x < 10) {
 if (x >= 0)
  x := x + 1;
 else
  x := x - 1;
}
assert(x <= 11);
```

Canonical Invariant	*Not−inv*	*Inv,not−ind*	*Inv,ind,not−adeq*	*Inv,ind,adeq*
$0 \leq x \leq 10$	$5 \leq x$	$-1 \leq x$	$0 \leq x \leq 12$	$0 \leq x \leq 11$



Handling nested loops



Verification conditions generated

Contracts for Recursive Functions

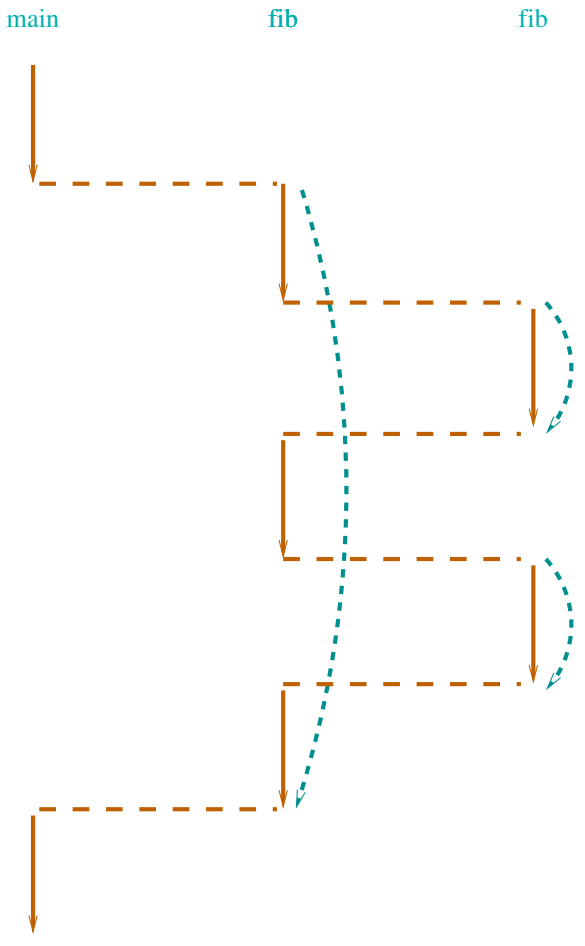
```
main() {
  result = fib(5);
  assert (result > 2);
}

// requires x >= 0
// ensures (result >= x) && (result > 0)
int fib(int x) {
  if (x < 2)
    return 1;
  else
    return fib(x-1) + fib(x-2);
}
```

For conjectured contract: $x \geq 0 \wedge \mathit{result} \geq x$, counterexample may be:

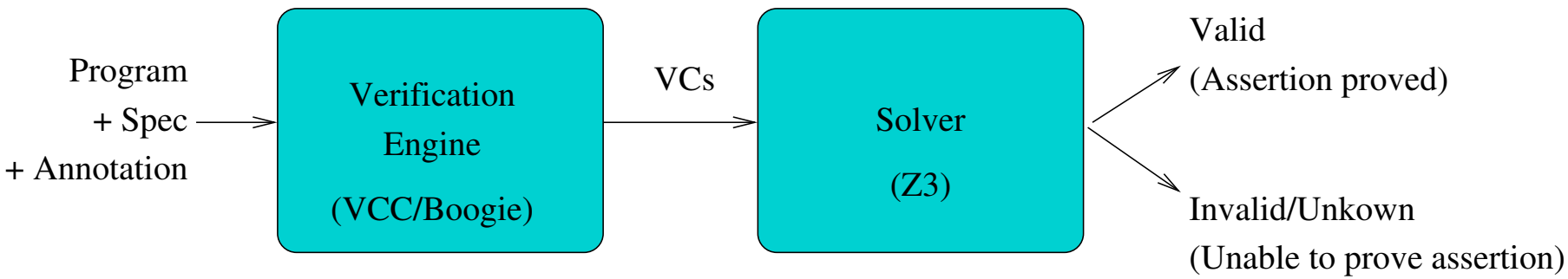
```
IF pre of fib contains a configuration with x=2 AND post contains a
configuration with (x=0, result=0) and another with (x=1, result=1),
THEN post must contain the configuration (x=2, result=1).
```

Soundness of Function Contracts

- Consider an execution of a program with valid contract annotation



How VCC Works (in Principle)



Conclusion

- Features of this Floyd-Hoare style of verification:
  - Tries to find a proof in the form of an inductive annotation.
  - A Floyd-style proof can be used to obtain a Hoare-style proof; and vice-versa.
  - Reduces verification (given key annotations) to checking satisfiability of a logical formula (VCs).
  - Is flexible about predicates, logic used (for example can add quantifiers to reason about arrays).
- Main challenge is the need for user annotation (adequate loop invariants).
- Can be increasingly automated (using learning techniques).