Zeroth-order logic

Text: Melvin Fitting, *FOLATP*, Sections 5.1,4.1,8.3
Homework: Read these sections, attempt exercises

Kamal Lodaya

February 2021

- In math, we use names like *Odd* and *Even* that are called predicates (in one variable). That is, $Odd(x)$ has a truth value if the variable x is assigned a number value.
- Over integers, $Odd(x)$ is *true* if x is assigned an odd value. Otherwise x is assigned an even value, so $Odd(x)$ is *false*.

Predicate logic with and without quantifiers

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- So we are saying what the language of our logic stands for, using our knowledge of numbers.

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- So we are saying what the language of our logic stands for, using our knowledge of numbers.
- Also want to say $Odd(2)$ is *false* and $Odd(-3)$ is *true*.
- In our syntax there are two kinds of symbols, predicates like *Odd*, *Even* and other things like $2, -3$.

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t &::= \quad x \in V \mid c \in C \mid f(t_1, \ldots, t_n),\ f \in F_n \\
A &::= \quad P(t_1, \ldots, t_n),\ P \in R_n \\
&\qquad \mid (\neg A) \mid (A \vee B) \mid (A \wedge B) \mid (A \supset B) \mid (A \equiv B)
\end{aligned}
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- Can construct parse tree for any well-formed formula (wff).
- For example, -3, short for $\mathit{minus}(3)$, is a unary function symbol *minus* applied to constant symbol 3.
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- $2 - 3$? Short for $add(2, minus(3))$, is binary function add applied to the trees for 2 and for -3.
- x^2 is short for $exp(x, 2)$ using a binary function symbol exp for exponentiation.
- Use standard math abbreviations, point them out first time.

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- Use standard math abbreviations, point them out first time.
- Terms are built up from variables and constant symbols using function symbols (eg, numbers with add, mul). The value of a term is in a domain of discourse such as $\mathbb{Z}$.
- A closed term is one without variables. Otherwise open.

Syntax of predicate logic (continued)

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& \mid \exists x A \mid \forall x A
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- Atomic formulas are built up from terms by applying an n-ary predicate symbol to n terms.
- For example, the binary predicates $<, \leq, >, \geq, =, \neq$ return boolean value.
  $2 < -3$ is short for $Lt(2, minus(3))$. Over $\mathbb{Z}$ returns *false*.

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  $2 < -3$ is short for $Lt(2, minus(3))$. Over $\mathbb{Z}$ returns *false*.
- $\approx$ is the equality symbol. For example,
  $(x + y)^2 \approx x^2 + 2xy + y^2$ is an abbreviated atomic formula.
  As usual, $2xy$ abbreviates $mul(2, mul(x, y))$.

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- Formulas are built up from atomic formulas using the boolean operations $\neg, \wedge, \vee, \supset, \equiv$ and quantifiers $(\forall x), (\exists x)$. A formula can only take a boolean value in $\{true, false\}$.
- For example, $(sqrt(2) < 0) \vee (sqrt(2) > 0)$ is a formula in our logic, so is $\neg(sqrt(2) < 0)$.
  Which of these formulas will be valid, evaluate to true?

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- For example, $(sqrt(2) < 0) \vee (sqrt(2) > 0)$ is a formula in our logic, so is $\neg(sqrt(2) < 0)$.
  Which of these formulas will be valid, evaluate to true?
- Is $x > y \supset \sqrt{x} > \sqrt{y}$ true? Jargon: $x > y$ is the antecedent of the implication, $\sqrt{x} > \sqrt{y}$ is the conclusion.
- A closed formula (or sentence) is one without variables. Otherwise it is an open formula.

Quantifiers or not? (Jacques Herbrand 1930)

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- Assume V, C, each F_n, each R_n, are finite or countable sets.
- Let $F = \bigcup_n F_n$, $R = \bigcup_n R_n$. These are countable sets.
- The signature of the logic is (R, F, C). For our examples, $R_1 = \{\mathit{Nat}, \mathit{Int}, \mathit{Real}\}$, $R_2 = \{<, \leq, >, \geq, , \neq\}$, $F_1 = \{\mathit{minus}, \mathit{sqrt}\}$, $F_2 = \{+, -, \times, /\}$, $C = \mathbb{Z}$.
- Quantifiers are not allowed in zeroth-order predicate logic (ZOL). Defined in text of Peter Andrews 1986.
- In first-order predicate logic (FOL), they are allowed. For the moment, stick to ZOL.

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Exercise

Are the set of terms and the set of formulas countable?

Types (Bertrand Russell 1908)

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- Going beyond our textbook, assume every term is typed. For example, constant 2 is in $\mathbb{N}$, constant -3 is in $\mathbb{Z}$, and term $\sqrt{5}$ is in $\mathbb{R}$. $\mathbb{N}$,$\mathbb{Z}$,$\mathbb{R}$ are called sorts or type symbols.
- Variables also come typed.
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- Variables also come typed.
- Can think of sorts as unary predicate symbols Nat, Int, $Real$. $Nat(x)$ is $true$ whenever x is in $\mathbb{N}$.
- More specifically, $sqrt$ is a unary function from $NonNegReal$ to $NonNegReal$.
  For the moment, assume $NonNegReal$ is a sort of ZOL.
  Later we can see how it is enforced as a signature.

Exercise

How does one type $(-b \pm \sqrt{b^2 - 4ac})/2a$ *in our term syntax?*

(Brahmagupta, Bhinmal, Jalore, Rajasthan, 7th century CE)

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- Yes. Proofs can be written in ZOL.
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- Case 1: It is a sentence and an axiom, which we accept as valid. For example, we accept $7 > 5$ as an axiom of the theory of natural numbers (or integers or reals) in the arithmetic signature defined earlier.

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- Case 1: It is a sentence and an axiom, which we accept as valid. For example, we accept $7 > 5$ as an axiom of the theory of natural numbers (or integers or reals) in the arithmetic signature defined earlier.
- Case 2: Such a sentence follows from earlier sentences in the sequence by application of an inference rule, which we accept as validity-preserving. (Square Root Monotonicity) is an inference rule for numbers of type *NonNegReal* in a theory, call it *Rplus*. It says that if $x > y$, then $\sqrt{x} > \sqrt{y}$.
- So here is a proof of a theorem (last line in a proof).

1	$7 > 5$	*Axiom*
2	$\sqrt{7} > \sqrt{5}$	1, *SqrtMon*

Derivations in a theory of nonnegative reals

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- A derivation, using inference rule (Addition Monotonicity). It says that if $x > y$, then $x + z > y + z$.

1	$3 > 2$	*Rplus*
2	$\sqrt{3} > \sqrt{2}$	1, *SqrtMon*
3	$\sqrt{7} > \sqrt{5}$	*Rplus*
4	$\sqrt{7} + \sqrt{3} > \sqrt{2} + \sqrt{5}$	2, 3, *AddMon*

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Question

Is $\sqrt{2} + \sqrt{7} > \sqrt{3} + \sqrt{5}$ a consequence of Rplus? What about $\sqrt{2} + \sqrt{11} < \sqrt{5} + \sqrt{7}$?

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4	$4\sqrt{14} > 3$	2, 3, *Repl*, *MP*

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6	$1 + 56 + 4\sqrt{14} > 60$	5, *Rplus*, *Repl*, *MP*

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7	$\sqrt{1 + 56 + 4\sqrt{14}} > \sqrt{60}$	6, *SqrtMon*
8	$1 + 2\sqrt{14} > 2\sqrt{15}$	7, *SE*, *Repl*, *Rplus*, *MP*

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9	$8+1+2\sqrt{14}>8+2\sqrt{15}$	8, *AddMon*
10	$2+7+2\sqrt{14}>3+5+2\sqrt{15}$	9, *Rplus*, *Repl*, *MP*

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10	$2 + 7 + 2\sqrt{14} > 3 + 5 + 2\sqrt{15}$	9, *Rplus*, *Repl*, *MP*
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- The Square Equation (SE) is the axiom scheme $(t+u)^2 \approx t^2 + 2tu + u^2$ in the arithmetic signature. It is a scheme because any terms t, u can be used. It is valid in *Rplus*, in fact over all reals.

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- The axiom schemes of Replacement (Repl) are $(t_1 \approx u_1) \wedge \cdots \wedge (t_n \approx u_n) \supset f(t_1, \ldots, t_n) \approx f(u_1, \ldots, u_n)$ and $(t_1 \approx u_1) \wedge \cdots \wedge (t_n \approx u_n) \supset (A(t_1, \ldots, t_n) \supset A(u_1, \ldots, u_n))$. They are schemes because any terms $t_1, \ldots, t_n, u_1, \ldots, u_n$, function symbol f and formula A can be used. Valid in all theories over all signatures, so they are ZOL axiom schemes.

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  They are schemes because any terms $t_1, \ldots, t_n, u_1, \ldots, u_n$, function symbol f and formula A can be used.
  Valid in all theories over all signatures, so they are ZOL axiom schemes.
- The inference rule Modus Ponens (affirming the antecedent) says that if the premisses A and $A \supset B$ both hold, then so does the conclusion B.
  Preserves validity in all theories over all signatures, so a ZOL inference rule.

Theory of numbers (Brahmagupta, Dedekind 1888)

- What is this theory *Rplus* of nonnegative reals?
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- Axioms for Dedekind's theory of numbers, going back to Brahmagupta's ring axioms:
- Addition and multiplication are commutative, associative.
  For example, $x + y \approx y + x, (x + y) + z \approx x + (y + z)$.
  They have units, for example, $x + 0 \approx x$.
  Multiplication distributes over addition, $x(y + z) \approx xy + xz$.
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Exercise

Give a proof of the Square Equation $(x + y)^2 \approx x^2 + 2xy + y^2$*, and Difference of Squares Equation* $x^2 - y^2 \approx (x + y)(x - y)$*, using Brahmagupta-Dedekind axioms.*

Problem

Characterize for natural numbers $a < b < c < d$, exactly when $\sqrt{a} + \sqrt{d} > \sqrt{b} + \sqrt{c}$.

- What did we gain by using logic?

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- What did we gain by using logic?
- In one word: to be precise.

Problem

Characterize for natural numbers $a < b < c < d$, exactly when $\sqrt{a} + \sqrt{d} > \sqrt{b} + \sqrt{c}$.

- What did we gain by using logic?
- In one word: to be precise.
- To be precise about signature and types, constants and variables, theories (surds vs reals), axioms and rules.
- How does it help to solve the problem above?

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Conjecture

Suppose for natural numbers $a < b < c < d$, the inequality $\sqrt{a} + \sqrt{d} > \sqrt{b} + \sqrt{c}$ is valid.
One can always find a proof for this in the theory of surds.

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- Obvious, left imprecise:
  When is a formula or sentence valid?
  When does a formula or sentence evaluate to *true*?