Theories and models

Text: Melvin Fitting, *FOLATP*, Sections 5.3,4.1,8.2
Homework: Read these sections, attempt exercises

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Syntax of ZOL over signature (R, F, C)

$$t ::= \quad x \in V \mid c \in C \mid f(t_1, \ldots, t_n),\ f \in F_n$$
$$A ::= \quad P(t_1, \ldots, t_n),\ P \in R_n \mid t_1 \approx t_2 \mid True \mid False$$
$$\mid (\neg A) \mid (A \vee B) \mid (A \wedge B) \mid (A \supset B) \mid (A \equiv B)$$

Quantifiers $\exists x A, \forall x A$ in FOL, not in ZOL

- Terms are built up from variables and constant symbols using function symbols (e.g., numbers with add, mul). The value of a term is in a domain of discourse such as $\mathbb{Z}$. Closed term is one without variables. Otherwise open.

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- Atomic formulas are built up from terms by applying an n-ary predicate symbol to n terms. Formulas are built up from atomic formulas using the boolean operations $\neg, \wedge, \vee, \supset, \equiv$. A formula gives a boolean value in $\{\mathit{true}, \mathit{false}\}$. Closed formula (or sentence) is one without variables. Otherwise it is an open formula.

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- Atomic formulas are built up from terms by applying an n-ary predicate symbol to n terms. Formulas are built up from atomic formulas using the boolean operations $\neg, \wedge, \vee, \supset, \equiv$. A formula gives a boolean value in $\{\mathit{true}, \mathit{false}\}$. Closed formula (or sentence) is one without variables. Otherwise it is an open formula.
- Every term is typed. Can think of basic types as unary predicate symbols like $\mathit{Nat}, \mathit{Int}, \mathit{Real}$.

Definition (Proof, theory, derivation)

A proof of a theorem A is a sequence of sentences ending with A. We write $\vdash A$. Each sentence is either an axiom, or following from earlier sentences in the sequence by application of an inference rule. A theory is a set of sentences.
A derivation of a consequence A from theory Th also allows members of Th in the sequence. We write $Th \vdash A$.
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Axiom schemes for equality (Gottfried Leibniz, 17th cent.CE):
(Reflexivity) $t \approx t$, (Replacement):
$(t_1 \approx u_1) \wedge \cdots \wedge (t_n \approx u_n) \supset f(t_1, \ldots, t_n) \approx f(u_1, \ldots, u_n)$
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Exercise (Equality theorems)

Transitivity $((x \approx y) \wedge (y \approx z)) \supset (x \approx z)$ and symmetry $x \approx y \supset y \approx x$ are theorems.

Proof system for ZOL (continued)

Inference rules:

(Modus Ponens/Detachment) If $\vdash A$ and $\vdash A \supset B$, then $\vdash B$.

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More axiom schemes:

(Positive paradox)	$A \supset (B \supset A)$
(Self distribution)	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (B \supset C))$
(False explosion)	$False \supset A, \quad A \supset True$
(Not elimination)	$\neg\neg A \supset A$
(Implicative explosion)	$A \supset (\neg A \supset B)$
(And elimination)	$(A \wedge B) \supset A, \quad (A \wedge B) \supset B$
(Or elimination)	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
(Iff elimination)	$(A \equiv B) \equiv ((A \supset B) \wedge (B \supset A))$

This ZOL proof system, with 13 axiom schemes and single inference rule (MP), is our base. From now on, a theory Th is assumed to already have all theorems of ZOL.

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Exercise (Contrapositive)

$(\neg B \supset \neg A) \equiv (A \supset B)$

Proof system BD, arithmetic signature $(+, -, \times, /, >)$

Axiom schemes (say over $\mathbb{Z}$) giving addition and multiplication tables to be added to ZOL:

$x + y \approx z,$ such that $x + y = z \in \mathbb{Z}$ $(e.g., 1 + 1 = 2)$

$x \times y \approx z,$ such that $xy = z \in \mathbb{Z}$ $(e.g., 14 \times 14 = 196)$

$x > y,$ such that $x = y + z$ for some $z > 0 \in \mathbb{Z}$ $(e.g., 224 > 9)$

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Axiom schemes from (Brahmagupta, Richard Dedekind):

$x + y \approx y + x, \quad (x + y) + z \approx x + (y + z)$
$(z \approx x - y) \equiv (x \approx y + z)$
$x \times y \approx y \times x, \quad (x \times y) \times z \approx x \times (y \times z)$
$x \times (y + z) \approx (x \times y) + (x \times z)$
$(z \approx x/y) \equiv (x \approx y \times z)$
$x < y \equiv y > x, \quad x \geq y \equiv (x > y \vee x \approx y)$

Call all these axiom schemes (with ZOL) the proof system BD.

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Exercise (Consequences of BD)

(Square Equation) $(x + y)^2 \approx x^2 + (2 \times x \times y) + y^2$

(Difference of Squares Equation) $x^2 - y^2 \approx (x + y) \times (x - y)$

(AddMon) $((x > y) \wedge (a \geq b)) \supset ((x + a) > (y + b))$

Surd signature $(+, -, \times, /, sqrt, >)$, nonnegative type

Inference rules from (Muhammad ibn Musa Al-Khwarizmi, Baghdad, 9th century CE) to be added to BD:

If $x \geq 0$, then $(\sqrt{x})^2 \approx \sqrt{x^2} \approx x$, $(\sqrt{x})(\sqrt{y}) \approx \sqrt{xy}$, $\frac{\sqrt{x}}{\sqrt{y}} \approx \sqrt{\frac{x}{y}}$

Call this proof system AK. Signature is larger, so more formulas and more theorems. More axioms, so even more theorems!

Exercise

$(x > 0) \supset (\sqrt{x} > 0)$, $\quad ((x > 0) \wedge (y > 0)) \supset (x/y > 0)$

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Derivation of (Sqrt Monotonicity) If $x > y$ then $\sqrt{x} > \sqrt{y}$ (math.stackexchange.com):

1	$x \geq 0, y \geq 0$	*Type premisses*
2	$x > y$	*Premiss*
3	$x - y > 0, x > 0$	2, *Simp*, *Trans*

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4	$\sqrt{y} \geq 0, \sqrt{x} > 0$	1, 2, *Exercise*, *MP*
5	$\sqrt{x} + \sqrt{y} > 0$	4, *AddMon*

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5	$\sqrt{x} + \sqrt{y} > 0$	4, *AddMon*
6	$(x - y)/(\sqrt{x} + \sqrt{y}) > 0$	2, 5, *Exercise*, *MP*
7	$\sqrt{x} - \sqrt{y} > 0$	6, *DSE*
8	$\sqrt{x} > \sqrt{y}$	7, *Simp*

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Definition

Model (also called structure) $M = (D, I)$, nonempty domain D, interpretation I which maps terms into the domain, as below. For c in C, $c^I \in D$; f in F_n, $f^I : D^n \to D$; P in R_n, $P^I \subseteq D^n$. □

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- Over the real domain $D = \mathbb{R}$, constant symbol π could be mapped to a number $3.14159265359\ldots$ (using math)
- A type, seen as a unary predicate, could be mapped $NonNegReal^I = \{d \in D \mid d \geq 0\}$
- No partial functions, must be properly typed.
  Since *sqrt* is typed $NonNegReal \to NonNegReal$, its interpretation would be
  $sqrt^I : NonNegReal^I \to NonNegReal^I$.

Definition

Assignment s *maps every* x *in* V *to* $x^s \in D$. □

- Lifting to terms:
  $x^{I,s} = x^s, \quad c^{I,s} = c^I, \quad f(t_1, \ldots, t_n)^{I,s} = f^I(t_1^{I,s}, \ldots, t_n^{I,s})$.

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- Interpretation I also maps every predicate symbol P in R_n to an n-ary relation $P^I \subseteq D^n$ over the domain.

  For example, over $\mathbb{R}$, $(-\sqrt{3}, \sqrt{2}) \in \leq^I$ and $(\sqrt{2}, -\sqrt{3}) \notin \leq^I$.

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  For example, over $\mathbb{R}$, $(-\sqrt{3}, \sqrt{2}) \in \leq^I$ and $(\sqrt{2}, -\sqrt{3}) \notin \leq^I$.
- Lifting to atomic formulas:
  $True^{I,s} = true, \quad False^{I,s} = false,$
  $P(t_1, \ldots, t_n)^{I,s} = true \text{ iff } (t_1^{I,s}, \ldots, t_n^{I,s}) \in P^I,$
  $(t_1 \approx t_2)^{I,s} = true \text{ iff } t_1^{I,s} = t_2^{I,s}.$

  The last condition defines a normal model, where equality is interpreted as equality, and not as some arbitrary equivalence relation.

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Formula evaluation in a ZOL model

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  For example,
  $\neg \mathit{true} = \mathit{false}$, $\mathit{false} \vee \mathit{true} = \mathit{true} = \mathit{true} \wedge \mathit{true}$,
  $\mathit{false} \supset \mathit{false} = \mathit{true}$, $\mathit{false} \supset \mathit{true} = \mathit{true}$, etc.

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  $\mathit{false} \supset \mathit{false} = \mathit{true}, \quad \mathit{false} \supset \mathit{true} = \mathit{true}$, etc.

- Lift to formulas. Left of equality is syntax, right is using table.

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\begin{aligned}
(\neg A)^{I,s} &= \neg(A^{I,s}) \\
(A \vee B)^{I,s} &= A^{I,s} \vee B^{I,s} \\
(A \wedge B)^{I,s} &= A^{I,s} \wedge B^{I,s} \\
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Formula satisfaction (alternative notation)

- Formula evaluation in interpretation I with assignment s, repeated from previous slide.

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Definition

For model $M = (D, I)$, assignment s, formula A satisfied when:

$M, s \models \neg A \text{ iff } M, s \not\models A$
$M, s \models A \vee B \text{ iff } M, s \models A \text{ or } M, s \models B$
$M, s \models A \wedge B \text{ iff } M, s \models A \text{ and } M, s \models B$
$M, s \models A \supset B \text{ iff } (\text{if } M, s \models A \text{ then } M, s \models B)$
$M, s \models A \equiv B \text{ iff } (M, s \models A \text{ iff } M, s \models B) \quad \square$

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$M, s \models \neg A$ iff $M, s \not\models A$

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$M, s \models A \wedge B$ iff $M, s \models A$ and $M, s \models B$

$M, s \models A \supset B$ iff (if $M, s \models A$ then $M, s \models B$)

$M, s \models A \equiv B$ iff ($M, s \models A$ iff $M, s \models B$) $\square$

Exercise

Check $(-\sqrt{3} \leq \sqrt{5}) \vee (\sqrt{5} \leq -\sqrt{3})$ evaluates to true/satisfied over $\mathbb{R}$ using typing.

What about $(x \leq y) \vee (y \leq x)$?

Formula validity and satisfiability

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Exercise (Duality)

Show that A is valid if and only if $\neg A$ is not satisfiable, and A is satisfiable if and only if $\neg A$ is not valid.