Theories to models

Text: Melvin Fitting, *FOLATP*, Sections 5.3,4.1,8.2,8.4
Homework: Read these sections, attempt exercises

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Evaluating terms in a Σ-model

$$t ::= \quad x \in V \mid c \in C \mid f(t_1, \ldots, t_n),\ f \in F_n$$
$$A ::= \quad P(t_1, \ldots, t_n),\ P \in R_n \mid t_1 \approx t_2 \mid True \mid False$$
$$\mid (\neg A) \mid (A \vee B) \mid (A \wedge B) \mid (A \supset B) \mid (A \equiv B)$$

Definition

Model (also called structure) $M = (D, I)$, nonempty domain D, interpretation I which maps terms into the domain, as below. For c in C, $c^I \in D$; f in F_n, $f^I : D^n \to D$; P in R_n, $P^I \subseteq D^n$. □

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- Over the real domain $D = \mathbb{R}$, constant symbol π could be mapped to a number $3.14159265359\ldots$ (using math)
- A type, seen as a unary predicate, could be mapped $NonNegReal^I = \{d \in D \mid d \geq 0\}$
- No partial functions, must be properly typed.
  Since $sqrt$ is typed $NonNegReal \to NonNegReal$, its interpretation would be
  $sqrt^I : NonNegReal^I \to NonNegReal^I$.

Definition

Assignment s *maps every* x *in* V *to* $x^s \in D$. □

- Lifting to terms:
  $x^{I,s} = x^s, \;\; c^{I,s} = c^I, \;\; f(t_1, \ldots, t_n)^{I,s} = f^I(t_1^{I,s}, \ldots, t_n^{I,s})$.

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- Lifting to atomic formulas:
  $True^{I,s} = true, \quad False^{I,s} = false,$
  $P(t_1, \ldots, t_n)^{I,s} = true \text{ iff } (t_1^{I,s}, \ldots, t_n^{I,s}) \in P^I,$
  $(t_1 \approx t_2)^{I,s} = true \text{ iff } t_1^{I,s} = t_2^{I,s}.$

  The last condition defines a normal model, where equality is interpreted as equality, and not as some arbitrary relation.

Exercise

What is the assignment for a model satisfying a sentence?

Definition

For model $M = (D, I)$, assignment s, formula A satisfied when:

$M, s \models \neg A$ iff $M, s \not\models A$

$M, s \models A \vee B$ iff $M, s \models A$ or $M, s \models B$

$M, s \models A \wedge B$ iff $M, s \models A$ and $M, s \models B$

$M, s \models A \supset B$ iff (if $M, s \models A$ then $M, s \models B$)

$M, s \models A \equiv B$ iff ($M, s \models A$ iff $M, s \models B$) $\square$

A formula A is M-valid ($M \models A$) if for every assignment s, $M, s \models A$. It is valid ($\models A$) if it is M-valid for all models M. Formula A is satisfiable if there is some model M where it is M-satisfiable, that is, for some assignment s, $M, s \models A$. $\square$

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Exercise

1. *Finish the above definition of satisfaction (atomic formulas).*
2. *Show that A is valid if and only if $\neg A$ is not satisfiable, and A is satisfiable if and only if $\neg A$ is not valid.*

Inference rule scheme:
(Modus Ponens/Detachment) If $\vdash A$ and $\vdash A \supset B$, then $\vdash B$.

Axiom schemes:

(Positive paradox)	$A \supset (B \supset A)$
(Self distribution)	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (B \supset C))$
(False explosion)	$\mathit{False} \supset A, \quad A \supset \mathit{True}$
(Not elimination)	$\neg\neg A \supset A$
(Implicative explosion)	$A \supset (\neg A \supset B)$
(And elimination)	$(A \wedge B) \supset A, \quad (A \wedge B) \supset B$
(Or elimination)	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
(Iff elimination)	$(A \equiv B) \equiv ((A \supset B) \wedge (B \supset A))$
(Reflexivity)	$t \approx t$

(Replacement):

$(t_1 \approx u_1) \wedge \cdots \wedge (t_n \approx u_n) \supset f(t_1, \ldots, t_n) \approx f(u_1, \ldots, u_n)$

$(t_1 \approx u_1) \wedge \cdots \wedge (t_n \approx u_n) \supset (A(t_1, \ldots, t_n) \supset A(u_1, \ldots, u_n))$

Theory BD, arithmetic signature $(+, -, \times, /, >)$

$x + y \approx z$, such that $x + y = z \in \mathbb{Z}$ $(e.g., 1 + 1 = 2)$
$x \times y \approx z$, such that $xy = z \in \mathbb{Z}$ $(e.g., 14 \times 14 = 196)$
$x > y$, such that $x = y + z$ for some $z > 0 \in \mathbb{Z}$ $(e.g., 224 > 9)$
$x + y \approx y + x, \quad (x + y) + z \approx x + (y + z)$
$(z \approx x - y) \equiv (x \approx y + z)$
$x \times y \approx y \times x, \quad (x \times y) \times z \approx x \times (y \times z)$
$x \times (y + z) \approx (x \times y) + (x \times z)$
$(z \approx x/y) \equiv (x \approx y \times z)$
$x < y \equiv y > x, \quad x \geq y \equiv (x > y \vee x \approx y)$

Theory AK, surd signature $(+, -, \times, /, sqrt, >)$, typed

If $x \geq 0$, then $(\sqrt{x})^2 \approx \sqrt{x^2} \approx x$, $(\sqrt{x})(\sqrt{y}) \approx \sqrt{xy}$, $\frac{\sqrt{x}}{\sqrt{y}} \approx \sqrt{\frac{x}{y}}$

For the arithmetic signature, all axiom schemes of proof system BD are $\mathbb{N}$-, $\mathbb{Z}$- and $\mathbb{R}$-valid with the usual interpretation for arithmetic operations. Inference rules of AK preserve validity for the surd signature over $\mathbb{R}$, that is, if the premisses are $\mathbb{R}$-valid, then so are the consequences.

Call this soundness of a proof system (over model M).
Formally, if $\vdash A$ then $\models A$,
and if $Th \vdash A$ for a theory over some signature, then $M \models A$ for models of that signature for which the theory was designed.

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For example, why is (Not elimination), $\neg\neg A \supset A$ valid?

Consider a model M, assignment s and $M, s \models \neg\neg A$.
By definition of satisfaction of negation, $M, s \not\models \neg A$.
Again by definition of satisfaction of negation, $M, s \models A$.

Exercise

Why is (False explosion), $False \supset A$ valid?

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Consider the integers with arithmetic modulo $n \geq 2$, written $\mathbb{Z}/n\mathbb{Z}$. Is the proof system BD sound over $\mathbb{Z}/n\mathbb{Z}$?

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Consider the integers with arithmetic modulo $n \geq 2$*, written* $\mathbb{Z}/n\mathbb{Z}$*. Is the proof system BD sound over* $\mathbb{Z}/n\mathbb{Z}$ *?*

If so, find an axiom which is not $\mathbb{Z}$*-valid but is* $\mathbb{Z}/n\mathbb{Z}$*-valid.*
If so, find another model over which BD is not sound.

Exercise

Consider the integers with arithmetic modulo $n \geq 2$, *written* $\mathbb{Z}/n\mathbb{Z}$. *Is the proof system BD sound over* $\mathbb{Z}/n\mathbb{Z}$ *?*

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Exercise (Difficult)

Consider $\mathbb{N}' = \mathbb{N} \cup \{\infty\}$, *with a new greatest element.*
Extend the operations in the expected way, for example let $x +^{\mathbb{N}'} \infty = \infty$. *Can BD be made sound over this* $\mathbb{N}'$ *?*

Definition

For theory Th, M is a model for Th if every sentence of Th is satisfied in M. (Th is an arbitrary set of sentences so one cannot write a conjunction.)
A formula is satisfiable modulo Th if it is M-satisfiable for some model M for Th. A formula is valid modulo Th if it is M-valid for all models M for Th. □

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- So far introduced formulas, then proofs and proof systems, then theories extending ZOL and corresponding proof systems, then models and corresponding models, validity and satisfiability.
- The purpose of the proof systems and theories is to formalize some mathematical ideas. (Database theories could be about database ideas.)
- What do we do with models?

(Herbrand 1930)'s term model

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Let Th be a deductively closed theory over signature Σ.

Definition (Herbrand model $M(Th)$ for Th)

Let $D \stackrel{\text{def}}{=} T(\Sigma)$ be a nonempty domain. Define the interpretation I on terms as follows: $c^I \stackrel{\text{def}}{=} c, \quad f^I(t_1, \ldots, t_n) \stackrel{\text{def}}{=} f(t_1^I, \ldots, t_n^I)$. (So this equals $f(t_1, \ldots, t_n)$ by induction on smaller terms.) Let $(t_1, \ldots, t_n) \in P^I$ iff $P(t_1, \ldots, t_n) \in Th$, making atomic formulas true exactly from Th. □

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- For example, for the arithmetic signature, could take all the axioms (not schemes) and theorems of proof system BD. Apart from $\mathbb{N}$,$\mathbb{Z}$,$\mathbb{R}$, another model over which BD is sound.

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Proposition (Exercise)

The Herbrand model $M(Th)$ is a model for Th.