Introducing propositional logic

Text: Melvin Fitting, *FOLATP*, Sections 8.3,8.4,2.2-2.4
Homework: Read these sections, attempt exercises

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Definition

For theory Th, M is a model for Th if every sentence of Th is satisfied in M. (Th is an arbitrary set of sentences so one cannot write a conjunction.)

A formula is satisfiable modulo Th if it is M-satisfiable for some model M for Th. A formula is valid modulo Th if it is M-valid for all models M for Th. □

(Jacques Herbrand 1930)'s term model

Consider a signature $\Sigma = (R, F, C)$ where C is nonempty. Then the set $T(\Sigma)$ of closed terms over the signature is nonempty.

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Let Th be a deductively closed theory over signature Σ.

Definition (Herbrand model $M(Th)$ for Th)

Let $D \stackrel{\text{def}}{=} T(\Sigma)$ be a nonempty domain. Define the interpretation I on terms as follows: $c^I \stackrel{\text{def}}{=} c, \quad f^I(t_1, \ldots, t_n) \stackrel{\text{def}}{=} f(t_1^I, \ldots, t_n^I)$.
(So this equals $f(t_1, \ldots, t_n)$ by induction on smaller terms.)
Let $(t_1, \ldots, t_n) \in P^I$ iff $P(t_1, \ldots, t_n) \in Th$. □

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Theorem (Satisfiability)

The Herbrand model $M(Th)$ is a model for the atomic sentences $P(t_1, \ldots, t_n)$ of Th.

Satisfaction (Hilbert-Bernays 1934, Tarski 1935)

Definition

For model $M = (D, I)$, assignment s, formula A satisfied when:

$M, s \models True$ (*always*)

$M, s \not\models False$ (*never*)

$M, s \models P(t_1, \ldots, t_n)$ iff $(t_1^I, \ldots, t_n^I) \in P^I$

$M, s \models \neg A$ iff $M, s \not\models A$

$M, s \models A \vee B$ iff $M, s \models A$ or $M, s \models B$

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- Suppose an undirected graph model with $Th = \{E(a, b), E(a, c), E(a, b) \supset E(b, a), E(a, c) \supset E(c, a)\}$. Then $M(Th) \models E(a, b) \wedge E(a, c) \wedge E(b, a) \wedge E(c, a)$.

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- Since Th deductively closed, it follows from an earlier Exercise that the relation $\approx^I$ in a Herbrand model is an equivalence relation.

Normal model construction (continued)

- By a standard "factor" construction, take blocks $[d] = d/\approx^I$ of every element d in the domain D as constituting a new domain $E = D/\approx^I$, and revise the interpretation I to a new interpretation J over the blocks.

  Thus $c^J = [c^I], \;\; f^J(t_1, \ldots, t_n) = [f(t_1, \ldots, t_n)^I]$ and $([t_1], \ldots, [t_n]) \in P^J \text{ iff } (t_1, \ldots, t_n) \in P^I$.

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- Because of the equivalence properties the choice of the term in a block does not matter. So $[t] \approx^J [u] \text{ iff } t \approx^I u \text{ iff } [t] = [u]$. So the new model (E, J) is normal. Show for all A, $(E, J), [d] \models A$ iff $(D, I), d \models A$. $\square$

In the example of the arithmetic signature, terms $1 + 1, 2$ map to the same element $[2]$, can be thought of as representing two.

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Exercise

Adding the equality axioms corresponds to restricting from all models to all normal models. Is it the case that adding an axiom to a proof system shrinks the class of models?

Introducing PL (Leibniz 17th c., 1704; Boole 1854)

ZOL syntax:

$$
\begin{aligned}
t ::= &\quad x \in V \mid c \in C \mid f(t_1, \ldots, t_n),\ f \in F_n \\
A ::= &\quad P(t_1, \ldots, t_n),\ P \in R_n \mid t_1 \approx t_2 \mid True \mid False \\
&\quad \mid (\neg A) \mid (A \vee B) \mid (A \wedge B) \mid (A \supset B) \mid (A \equiv B)
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Propositional logic PL over At has simpler syntax. Atomic formulas in At are called propositional letters. No terms.

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- A valuation v (also called propositional model) assigns a boolean value $v(P)$ to every propositional letter P in At.
- This is lifted to formulas just as was done before, by Table 2.1 in (Fitting, Section 2.3, page 13).
- A formula A is satisfiable if there is some valuation v such that $v(A) = true$ through this lifting. A formula is a tautology/valid if it evaluates to $true$ under all valuations.
- Th is satisfiable if some valuation v makes all A in Th true.

Proof system for PL, (Aristotle, 4th c.BCE) onwards

Inference rule scheme:
(Modus Ponens/Detachment) If $\vdash A$ and $\vdash A \supset B$, then $\vdash B$.

Axiom schemes:

(Positive paradox)	$A \supset (B \supset A)$
(Self distribution)	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (B \supset C))$
(False explosion)	$False \supset A, \quad A \supset True$
(Not elimination)	$\neg\neg A \supset A$
(Implicative explosion)	$A \supset (\neg A \supset B)$
(And elimination)	$(A \wedge B) \supset A, \quad (A \wedge B) \supset B$
(Or elimination)	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
(Iff elimination)	$(A \equiv B) \equiv ((A \supset B) \wedge (B \supset A))$

This proof system is sound for Boolean valuations, which are models for PL.

Propositional abstraction of ZOL

$$A ::= P,\ P \in At \mid True \mid False$$
$$\mid (\neg A) \mid (A \vee B) \mid (A \wedge B) \mid (A \supset B) \mid (A \equiv B)$$

Assume E is not a predicate symbol in ZOL signature Σ. Define $At(\Sigma) = \{(P, t_1, \ldots, t_n), (E, t_1, t_2) \mid P \in R_n, t_i \in T(\Sigma), i \in [1, n]\}$.

- That is, every equation and every predicate instance correspond to a unique propositional letter in set $(\{E\} \times T(\Sigma) \times T(\Sigma)) \cup \bigcup_{n>0,\ P \in R_n} (\{P\} \times T(\Sigma)^n)$.
- Can be thought of as mapping expanded signature $\Sigma \cup At(\Sigma)$, where new letters are thought of as 0-ary predicate symbols, to smaller signature $At(\Sigma)$.

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- Can be thought of as mapping expanded signature $\Sigma \cup At(\Sigma)$, where new letters are thought of as 0-ary predicate symbols, to smaller signature $At(\Sigma)$.
- Define an uninterpreting translation from formulas of signature Σ to formulas over the letters $At(\Sigma)$.

$$Un(P(t_1, \ldots, t_n)) = (P, t_1, \ldots, t_n),\ Un(t_1 \approx t_2) = (E, t_1, t_2),$$
$$Un(True) = True,\ Un(False) = False,\ Un(\neg A) = \neg Un(A),$$
$$Un(A \circ B) = Un(A) \circ Un(B),\ \text{for } \circ \in \{\vee, \wedge, \supset, \equiv\}$$

This leads to a “free” construction. Such constructions were done by (Leopold Kronecker 1887) in algebra.

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- Define over Σ theory $Th = \{A \mid v(Un(A)) = true\}$. So those atomic formulas occurring in A which v made $true$ are in Th, those which it made $false$ are not in Th.

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- Define over Σ theory $Th = \{A \mid v(Un(A)) = true\}$. So those atomic formulas occurring in A which v made $true$ are in Th, those which it made $false$ are not in Th.
- Let M be the Herbrand model for Th, satisfying precisely the atomic sentences which are in Th. Satisfaction for ZOL lifts to formulas using the same truth table as for PL, homomorphically. So $M \models A$ as well. □

Definition (Fitting, Definition 3.5.1,8.4.5)

Th is a Hintikka theory if it is downwards closed.

1. $False, \neg True \notin Th$; for P in At, $\{P, \neg P\} \not\subseteq Th$
2. If $\neg\neg A$ in Th, then A in Th
3. If $A \wedge B$ in Th, then $\{A, B\} \subseteq Th$
4. If $A \vee B$ in Th, then either A in Th or B in Th
5. If $\neg(A \wedge B)$ in Th, then either $\neg A$ in Th or $\neg B$ in Th
6. If $\neg(A \vee B)$ in Th, then $\{\neg A, \neg B\} \subseteq Th$

Theorem (PL/ZOL sat, Fitting, Proposition 3.5.2, Page 52)
PL and ZOL Hintikka theories Th are satisfiable.

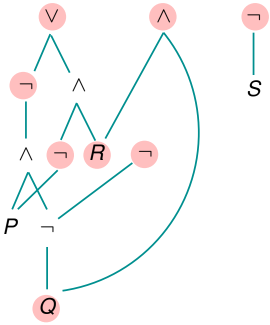
Proof.
For a ZOL Hintikka theory, use the earlier theorems to reduce to PL.

The PL Hintikka theory conditions are used to downward close the theory. This means that all the propositional letters used in a definition of satisfaction of the theory are inside the theory.

Consider valuation v where $v(P) = true$ iff $P \in Th$, this is well-defined by condition (1) in the definition on the last slide.

Verify all the conditions in the definition of satisfaction. $\square$

Consider the theory H:
$\{(\neg(P \wedge \neg Q)) \vee ((\neg P) \wedge R), \neg(P \wedge \neg Q), \neg\neg Q, Q, R \wedge Q,\ R, \neg S\}$.



Satisfying valuation is obtained by setting propositions in H to *true*, and the rest to *false*:
$v : P \mapsto \mathit{false}, Q \mapsto \mathit{true}, R \mapsto \mathit{true}, S \mapsto \mathit{false}$.

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- Mathematical logic finds general constructions of models for specifications/theories.
- Introducing ZOL (most books start with PL and then do FOL) was to bring some constructions out from long proofs (Terence Tao, blog 2009).