Consistency

Text: Melvin Fitting, *FOLATP*, Sections 3.4,3.5,4.1
Homework: Read these sections, attempt exercises

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Hilbert-Bernays proof in system BD

1	$224 > 9$	*Rplus*
2	$\sqrt{224} > \sqrt{9}$	1, *SqrtMon*
3	$4\sqrt{14} \approx \sqrt{224},\ 3 \approx \sqrt{9}$	*Rplus*
4	$4\sqrt{14} > 3$	2, 3, *Repl*, *MP*
5	$57 + 4\sqrt{14} > 57 + 3$	4, *AddMon*
6	$1 + 56 + 4\sqrt{14} > 60$	5, *Rplus*, *Repl*, *MP*
7	$\sqrt{1 + 56 + 4\sqrt{14}} > \sqrt{60}$	6, *SqrtMon*
8	$1 + 2\sqrt{14} > 2\sqrt{15}$	7, *SE*, *Repl*, *Rplus*, *MP*
9	$8 + 1 + 2\sqrt{14} > 8 + 2\sqrt{15}$	8, *AddMon*
10	$2 + 7 + 2\sqrt{14} > 3 + 5 + 2\sqrt{15}$	9, *Rplus*, *Repl*, *MP*
11	$\sqrt{2 + 7 + 2\sqrt{14}} > \sqrt{3 + 5 + 2\sqrt{15}}$	10, *SqrtMon*
12	$\sqrt{2} + \sqrt{7} > \sqrt{3} + \sqrt{5}$	11, *SE*, *Repl*, *MP*

How does one find such a proof?

Proof by contradiction

1	$[\sqrt{2} + \sqrt{7} \leq \sqrt{3} + \sqrt{5}]$	*Negated*
2	$[2 + 7 + 2\sqrt{14} \leq 3 + 5 + 2\sqrt{15}]$	1, *SqrtMon*
3	$[1 + 2\sqrt{14} \leq 2\sqrt{15}]$	2, *Simp*
4	$[57 + 4\sqrt{14} \leq 60]$	3, *SqrtMon*
5	$[4\sqrt{14} \leq 3]$	4, *Simp*
6	$[\sqrt{224} \leq \sqrt{9}]$	5, *Simp*
7	$[224 \leq 9]$	6, *SqrtMon*
8	[]	*Rplus*

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8	[]	*Rplus*

Think of every line of the proof as being satisfiable. Application of a rule preserves satisfiability.

1	$[\sqrt{2} + \sqrt{7} \leq \sqrt{3} + \sqrt{5}]$	*Negated*
2	$[2 + 7 + 2\sqrt{14} \leq 3 + 5 + 2\sqrt{15}]$	1, *SqrtMon*
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5	$[4\sqrt{14} \leq 3]$	4, *Simp*
6	$[\sqrt{224} \leq \sqrt{9}]$	5, *Simp*
7	$[224 \leq 9]$	6, *SqrtMon*
8	$[]$	*Rplus*

Think of every line of the proof as being satisfiable. Application of a rule preserves satisfiability.

On the other hand the empty clause (*False*) is unsatisfiable. Tracing backwards preserves unsatisfiability, so $\neg A$ is unsatisfiable, that is, A is valid.

Definition (Fitting, Definition 3.5.1,8.4.5)

Th is a Hintikka theory if it is downwards closed.

1. $False, \neg True \notin Th$; for P in At, $\{P, \neg P\} \not\subseteq Th$
2. If $\neg\neg A$ in Th, then A in Th
3. If $A \wedge B$ in Th, then $\{A, B\} \subseteq Th$
4. If $A \vee B$ in Th, then either A in Th or B in Th
5. If $A \supset B$ in Th, then either $\neg A$ in Th or B in Th
6. If $\neg(A \wedge B)$ in Th, then either $\neg A$ in Th or $\neg B$ in Th
7. If $\neg(A \vee B)$ in Th, then $\{\neg A, \neg B\} \subseteq Th$
8. If $\neg(A \supset B)$ in Th, then $\{A, \neg B\} \subseteq Th$

Exercise

Extend $Th = \{((P \supset Q) \vee ((\neg P) \wedge R)), \neg\neg Q, Q \wedge R, \neg S\}$ to a Hintikka theory and find a valuation to satisfy it.

Proof system for PL (Hilbert and Bernays 1918)

Developed from *Principia Mathematica* (Alfred Whitehead and Bertrand Russell 1910).

Inference rule scheme:
(Modus Ponens/Detachment) If $\vdash A$ and $\vdash A \supset B$, then $\vdash B$.

Axiom schemes:

(Positive paradox)	$A \supset (B \supset A)$
(Self distribution)	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (B \supset C))$
(False explosion)	$\mathit{False} \supset A, \quad A \supset \mathit{True}$
(Not elimination)	$\neg\neg A \supset A$
(Implicative explosion)	$A \supset (\neg A \supset B)$
(And elimination)	$(A \wedge B) \supset A, \quad (A \wedge B) \supset B$
(Or elimination)	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
(Iff elimination)	$(A \equiv B) \equiv ((A \supset B) \wedge (B \supset A))$

This proof system is sound for Boolean valuations.

Definition

Let Th a theory and A a formula.

A is a consequence of Th if $Th \vdash A$.

Th is inconsistent if $False$ is a consequence of Th.

Th is consistent if it is not inconsistent, that is, $Th \nvdash False$.

Exercise

Show that Th is consistent iff every finite subset of Th is consistent.

Hint: Equivalent to showing that Th is inconsistent iff some finite subset of Th is inconsistent.