Completeness

Text: Melvin Fitting, *FOLATP*, Sections 3.5,4.1
Homework: Read these sections, attempt exercises

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$$A ::= P,\ P \in At \mid True \mid False \mid (\neg A) \mid (A \vee B) \mid (A \wedge B) \mid (A \supset B) \mid (A \equiv B)$$

Definition

A Boolean model M is a valuation $v : At \to \{true, false\}$.
Formula A is satisfied when:

$M \models True$ (*always*)

$M \not\models False$ (*never*)

$M \models P$ iff $v(P) = true$

$M \models \neg A$ iff $M \not\models A$

$M \models A \vee B$ iff $M \models A$ or $M \models B$

$M \models A \wedge B$ iff $M \models A$ and $M \models B$

$M \models A \supset B$ iff (if $M \models A$ then $M \models B$)

$M \models A \equiv B$ iff ($M \models A$ iff $M \models B$)

A is valid iff for all models M, $M \models A$. A is satisfiable iff for some model M, $M \models A$. □

- Thus validity and satisfiability are dual notions.

PL proof system (Frege 1879, Hilbert-Bernays 1918)

Inference rule scheme:
(Modus Ponens/Detachment) If $\vdash A$ and $\vdash A \supset B$, then $\vdash B$.

Axiom schemes:

(Positive paradox)	$A \supset (B \supset A)$
(Self distribution)	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$[1]
(False explosion)	$False \supset A, \quad A \supset True$ (True implosion)
(Not elimination)	$\neg\neg A \supset A$
(Implicative explosion)	$A \supset (\neg A \supset B)$
(And elimination)	$(A \wedge B) \supset A, \quad (A \wedge B) \supset B$
(Or elimination)	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
(Iff elimination)	$(A \equiv B) \equiv ((A \supset B) \wedge (B \supset A))$

- This proof system is sound for Boolean valuations. Every axiom is valid, and the inference rule preserves validity.
- Is it complete? That is, suppose A is valid. Can we find a proof for A using the proof system?

[1] There was a mistake in earlier slides, corrected.

How does one find proofs?
One idea is to do it by contradiction.

Definition

Let Th be a PL theory and A be a PL formula.

- *A is a consequence of Th if $Th \vdash A$ (more correctly pHB-consequence, $Th \vdash_{pHB} A$).*
- *Th is (pHB-)inconsistent if $False$ (pHB-)consequence of Th.*
- *Th is (pHB-)consistent if it is not (pHB-)inconsistent, that is, $Th \not\vdash_{pHB} False$.*

- $\{P\}$ is consistent (write P is consistent), not obvious.
- $\{P, P \supset (Q_1 \wedge Q_2), \bigwedge_{i=1}^{2} Q_i \supset (R_{i1} \wedge R_{i2}), R_{22} \supset \neg P\}$ incons.
- How to check consistency?

Deduction theorem (Jacques Herbrand 1930)

Theorem (Deduction, Fitting, Theorem 4.1.4)

For theory Th, formulas A, B and a Hilbert-Bernays proof system with (Positive paradox), (Self-distribution) and single inference rule (MP), $Th \cup \{A\} \vdash B$ iff $Th \vdash (A \supset B)$.

Proof. ($\Longleftarrow$) (MP).

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Proof. ($\Longleftarrow$) (MP).

($\Longrightarrow$) Suppose $Z_1, \ldots, Z_n$ is a derivation of $Th \cup \{A\} \vdash B$.
To show that: $A \supset Z_1, \ldots, A \supset Z_n$ is a derivation of $A \supset B$.

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If Z_i is an axiom or a premiss but not A,
(Positive paradox) axiom $Z_i \supset (A \supset Z_i)$ gives a proof.

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If Z_i is A, insert a proof of $A \supset A$ (Fitting, Section 4.1, page 72).

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If Z_i is A, insert a proof of $A \supset A$ (Fitting, Section 4.1, page 72).

If Z_i is derived from earlier Z_j and $Z_j \supset Z_i$, (Self-distribution) axiom $(A \supset (Z_j \supset Z_i)) \supset ((A \supset Z_j) \supset (A \supset Z_i))$ gives proof. □

Exercise (Duality)

Show that $\vdash A$ iff $\neg A$ is inconsistent and A is consistent iff $\nvdash \neg A$.

Definition (Fitting, Definition 3.5.1,8.4.5)

A Hintikka theory Th (also called downwards consistent) has the following easy conditions to avoid inconsistency.

1. $False, \neg True \notin Th$; for P in At, $\{P, \neg P\} \not\subseteq Th$
2. If $\neg\neg A$ in Th, then A in Th
3. If $A \wedge B$ in Th, then $\{A, B\} \subseteq Th$
4. If $A \vee B$ in Th, then either A in Th or B in Th
5. If $A \supset B$ in Th, then either $\neg A$ in Th or B in Th
6. If $A \equiv B$ in Th, then $\{A \supset B, B \supset A\} \subseteq Th$
7. If $\neg(A \wedge B)$ in Th, then either $\neg A$ in Th or $\neg B$ in Th
8. If $\neg(A \vee B)$ in Th, then $\{\neg A, \neg B\} \subseteq Th$
9. If $\neg(A \supset B)$ in Th, then $\{A, \neg B\} \subseteq Th$
10. If $\neg(A \equiv B)$ in Th, then $\neg(A \supset B)$ in Th or $\neg(B \supset A)$ in Th

- Hintikka theories can be thought of as providing as much of a "state description" as possible.
- $Th = \{P, P \supset (Q_1 \wedge Q_2), \bigwedge_{i=1}^{2} Q_i \supset (R_{i1} \wedge R_{i2}), R_{22} \supset \neg P\}$.

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- In many steps reach $Th \cup \{P, Q_1, Q_2, R_{11}, R_{12}, R_{21}, R_{22}\}$.
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- One more step will add $\neg P$ revealing $\{P, \neg P\}$.
- On the other hand, if a theory is consistent, downwards closure will not make it inconsistent. (Next theorem.) Not clear if this is sufficient.
- A related idea of "maximal" theories was used by Adolf Lindenbaum in his lectures, credited in Alfred Tarski 1930, published after Lindenbaum's murder during World War 2.

Consistent theories and Hintikka conditions

Lemma

For consistent PL theory Th, $False, \neg True \notin Th$, $\{P, \neg P\} \not\subseteq Th$.

Proof of one case.

To show $\neg True \notin Th$, suppose for contradiction that $\neg True \in Th$, so $Th \vdash \neg True$. By (True implosion), $\neg True \supset True$, $Th \vdash True$ by (MP). By (Implicative explosion), $True \supset (\neg True \supset False)$, $Th \vdash False$ by (MP), which contradicts consistency of Th. □

Theorem (Fitting, Lemma 4.1.7)

A finite consistent PL theory Th can be extended to a Hintikka theory.

- Each violation of a Hintikka condition is rectified by extending the theory maintaining consistency.
- Termination uses König's lemma (Fitting, Theorem 2.7.2).

Proof of one case.
Assume $A \vee B$ in Th, that is, $Th \cup \{A \vee B\}$ is consistent. Suppose for contradiction that $Th \cup \{A\} \vdash False$ and $Th \cup \{B\} \vdash False$ are inconsistent. By the Deduction Theorem, $Th \vdash A \supset False$ and $Th \vdash B \supset False$. By (Or-elimination), $Th \vdash (A \vee B) \supset False$. By the Deduction Theorem, $Th \cup \{A \vee B\} \vdash False$, which contradicts consistency of our hypothesis. So either $Th \cup \{A\}$ or $Th \cup \{B\}$ is a consistent extension containing A or B. $\square$

Consistent theories to models

Earlier in Lecture 5 (Intro PL) we showed:

Theorem (PL sat, Fitting, Proposition 3.5.2, Page 52)
Hintikka PL theories are satisfiable in a PL model.

- In the proof of this theorem item (1) of the Hintikka theory definition was used.
- So, putting the two results together, finite consistent PL theories are satisfiable.

Theorem (Completeness)

A valid PL formula is provable in the pHB proof system.

Proof.

Using duality of validity and satisfiability, and duality of theoremhood and consistency, this is equivalent to proving that a pHB-consistent PL formula is satisfiable.

Contrapositively, assume A is not provable. By (Not-elimination), $\nvdash \neg\neg A$. To be shown that A is not valid.

By Exercise, $\neg A$ is consistent. By the result of the previous slide, $\neg A$ is satisfiable. This means A is not a valid formula. □

Soundness and completeness

- The other direction was soundness (theorem implies valid formula, or contrapositively, satisfiable implies consistent). So we have soundness and completeness theorems.
- Alternately, from the proofs seen above:
  PL formula is satisfiable iff it is consistent iff it is a member of a Hintikka PL theory.
- So extending to Hintikka theory is sufficient to check consistency.