Decidability

Text: Melvin Fitting, *FOLATP*, Sections 3.4,4.4,4.5
Homework: Read these sections, attempt exercises

Kamal Lodaya

March 2021

- There are many different proof systems. Our proof systems are weakly sound, every theorem is a valid formula. Strongly sound, every consequence of a theory is valid modulo that theory. Weakly complete, every valid formula [1] has a proof in that proof system (Emil Post 1921).
- Strongly complete, every valid formula modulo Th ($Th \models A$) has a derivation from Th ($Th \vdash A$). Proved in textbook.

Question

Algorithm which says whether a formula is valid?

[1] "Tautology" is from *Tractatus logico-philosophicus* (Wittgenstein 1921), came up with truth tables independently of Post. Predated by (Peirce 1893).

- There are many different proof systems. Our proof systems are weakly sound, every theorem is a valid formula. Strongly sound, every consequence of a theory is valid modulo that theory. Weakly complete, every valid formula [1] has a proof in that proof system (Emil Post 1921).
- Strongly complete, every valid formula modulo Th ($Th \models A$) has a derivation from Th ($Th \vdash A$). Proved in textbook.

Question

Algorithm which says whether a formula is valid?

- Consistent theories are the counterpart of satisfiability. They extend to Hintikka theories, then models constructed.
- Martin Davis, Hilary Putnam, George Logemann and Donald Loveland 1960-61 came up with a test for propositional satisfiability. Basis for modern sat solvers.

[1] "Tautology" is from *Tractatus logico-philosophicus* (Wittgenstein 1921), came up with truth tables independently of Post. Predated by (Peirce 1893).

A refutation system

1 $[\neg((a \approx b) \wedge (b \approx c) \supset a \approx c)]$ *Negated*

1	$[\neg((a \approx b) \wedge (b \approx c) \supset a \approx c)]$	*Negated*
$2a$	$[(a \approx b) \wedge (b \approx c)]$	
$2b$	$[\neg(a \approx c)]$	1, *NotImplies*

1	$[\neg((a \approx b) \wedge (b \approx c) \supset a \approx c)]$	*Negated*
$2a$	$[(a \approx b) \wedge (b \approx c)]$	
$2b$	$[\neg(a \approx c)]$	1, *NotImplies*
$3a$	$[a \approx b]$	
$3b$	$[b \approx c]$	$2a$, *And*

1	$[\neg((a \approx b) \wedge (b \approx c) \supset a \approx c)]$	*Negated*
$2a$	$[(a \approx b) \wedge (b \approx c)]$	
$2b$	$[\neg(a \approx c)]$	1, *NotImplies*
$3a$	$[a \approx b]$	
$3b$	$[b \approx c]$	$2a$, *And*
4	$[\neg(b \approx c)]$	$2b, 3a$, *Repl*

1	$[\neg((a \approx b) \wedge (b \approx c) \supset a \approx c)]$	*Negated*
$2a$	$[(a \approx b) \wedge (b \approx c)]$	
$2b$	$[\neg(a \approx c)]$	1, *NotImplies*
$3a$	$[a \approx b]$	
$3b$	$[b \approx c]$	$2a$, *And*
4	$[\neg(b \approx c)]$	$2b, 3a$, *Repl*
5	$[]$	$3b, 4$, *Cut*

Definition

- *A literal L is an atomic formula (positive) or its negation. Its complement is $\overline{L}$: $\overline{P} = \neg P$, $\overline{\neg P} = P$.*
- *A clause is a disjunction $C = [L_1, \ldots, L_n]$ where every L_i is a literal. Empty clause $[]$ is False.*
- *A PL formula in conjunctive normal form (CNF) is a clause set $S = \langle C_1; \ldots; C_m \rangle$ where each C_i is a clause. Empty clause set $\langle\rangle$ is True.*
- *A block is a disjunction of clause sets $[S_1, \ldots, S_l]$.*

Conjunctive normal form rules

Theorem

There are algorithms for converting a formula into CNF.

Proof.

(Fitting, Table 2.3, page 26) gives rewrite rules, including:

$[\ldots, \neg\neg A, \ldots] \rightarrow [\ldots, A, \ldots]$, $[\ldots, A \vee B, \ldots] \rightarrow [\ldots, A, B, \ldots]$,
$[\ldots, A \supset B, \ldots] \rightarrow [\ldots, \neg A, B, \ldots]$,
$[\ldots, \neg(A \wedge B), \ldots] \rightarrow [\ldots, \neg A, \neg B, \ldots]$,

Theorem

There are algorithms for converting a formula into CNF.

Proof.

(Fitting, Table 2.3, page 26) gives rewrite rules, including:

$[\ldots, \neg\neg A, \ldots] \to [\ldots, A, \ldots]$, $[\ldots, A \vee B, \ldots] \to [\ldots, A, B, \ldots]$,
$[\ldots, A \supset B, \ldots] \to [\ldots, \neg A, B, \ldots]$,
$[\ldots, \neg(A \wedge B), \ldots] \to [\ldots, \neg A, \neg B, \ldots]$,
$[\ldots, A \wedge B, \ldots] \to \langle [\ldots, A, \ldots]; [\ldots, B, \ldots] \rangle$,
$[\ldots, \neg(A \vee B), \ldots] \to \langle [\ldots, \neg A, \ldots]; [\ldots, \neg B, \ldots] \rangle$,
$[\ldots, \neg(A \supset B), \ldots] \to \langle [\ldots, A, \ldots]; [\ldots, \neg B, \ldots] \rangle$.

The last three rules rewrite to conjunctions of disjunctions.

Applying each rewrite rule, called CNF expansion, leads to an equivalent formula. Termination: only literals. □

Exercise

Use König's lemma to prove termination.

Exercise

1. *Given a truth table evaluation for a formula A under all possible valuations of its letters, write equivalent CNF for A.*
2. *Convert $(P_1 \wedge Q_1) \vee (P_2 \wedge Q_2) \vee \cdots \vee (P_n \wedge Q_n)$ to CNF. Show that naively applied, conversion to CNF may blow up formula size exponentially.*

Conversion to conjunctive normal form

Exercise

1. *Given a truth table evaluation for a formula A under all possible valuations of its letters, write equivalent CNF for A.*
2. *Convert $(P_1 \wedge Q_1) \vee (P_2 \wedge Q_2) \vee \cdots \vee (P_n \wedge Q_n)$ to CNF. Show that naively applied, conversion to CNF may blow up formula size exponentially.*
3. *(Tseitin 1968) transformation: using fresh letters $R_1, \ldots, R_n$, compare satisfiability of previous formula with:*

$$(R_1 \vee \cdots \vee R_n) \wedge ((R_1 \supset (P_1 \wedge Q_1)) \wedge ((R_2 \supset (P_2 \wedge Q_2)) \wedge \cdots \wedge (R_n \supset (P_n \wedge Q_n))$$

Theorem (Richard Karp 1972)

There is a polynomial time reduction from satisfiability to satisfiability in CNF.

Cut (Gerhard Gentzen 1934, Alan Robinson 1968)

- Cut/Resolution on A: $\langle[B, A]; [\neg A, C]\rangle \rightarrow [B, C]$. From (MP).
- One-literal/Unit cut propagation: If $[L]$ in clause set S, delete clauses of S containing L (including unit clause) and occurrences in S of complement $\overline{L}$.

1 $[\neg(((P \wedge Q) \vee (R \supset S)) \supset ((P \vee (R \supset S)) \wedge (Q \vee (R \supset S))))]$

Cut (Gerhard Gentzen 1934, Alan Robinson 1968)

- Cut/Resolution on A: $\langle[B, A]; [\neg A, C]\rangle \to [B, C]$. From (MP).
- One-literal/Unit cut propagation: If $[L]$ in clause set S, delete clauses of S containing L (including unit clause) and occurrences in S of complement $\overline{L}$.

$$
\begin{array}{lll}
1 & [\neg(((P \wedge Q) \vee (R \supset S)) \supset ((P \vee (R \supset S)) \wedge (Q \vee (R \supset S))))] & \\
2a & [P \wedge Q, R \supset S] & \\
2b & [\neg((P \vee (R \supset S)) \wedge (Q \vee (R \supset S)))] & 1, \mathit{NotImplies} + \mathit{Or}
\end{array}
$$

Cut (Gerhard Gentzen 1934, Alan Robinson 1968)

- Cut/Resolution on A: $\langle[B, A]; [\neg A, C]\rangle \to [B, C]$. From (MP).
- One-literal/Unit cut propagation: If $[L]$ in clause set S, delete clauses of S containing L (including unit clause) and occurrences in S of complement $\overline{L}$.

1	$[\neg(((P \wedge Q) \vee (R \supset S)) \supset ((P \vee (R \supset S)) \wedge (Q \vee (R \supset S))))]$	
$2a$	$[P \wedge Q, R \supset S]$	
$2b$	$[\neg((P \vee (R \supset S)) \wedge (Q \vee (R \supset S)))]$	$1, \mathit{NotImplies} + \mathit{Or}$
3	$[P, R \supset S]; [Q, R \supset S]$	$2a, \mathit{And}$

Cut (Gerhard Gentzen 1934, Alan Robinson 1968)

- Cut/Resolution on A: $\langle [B, A]; [\neg A, C] \rangle \rightarrow [B, C]$. From (MP).
- One-literal/Unit cut propagation: If $[L]$ in clause set S, delete clauses of S containing L (including unit clause) and occurrences in S of complement $\overline{L}$.

1	$[\neg(((P \wedge Q) \vee (R \supset S)) \supset ((P \vee (R \supset S)) \wedge (Q \vee (R \supset S))))]$	
$2a$	$[P \wedge Q, R \supset S]$	
$2b$	$[\neg((P \vee (R \supset S)) \wedge (Q \vee (R \supset S)))]$	$1, NotImplies + Or$
3	$[P, R \supset S]; [Q, R \supset S]$	$2a, And$
4	$[\neg(P \vee (R \supset S)), \neg(Q \vee (R \supset S))]$	$2b, NotAnd$

Cut (Gerhard Gentzen 1934, Alan Robinson 1968)

- Cut/Resolution on A: $\langle [B, A]; [\neg A, C] \rangle \rightarrow [B, C]$. From (MP).
- One-literal/Unit cut propagation: If $[L]$ in clause set S, delete clauses of S containing L (including unit clause) and occurrences in S of complement $\overline{L}$.

1	$[\neg(((P \wedge Q) \vee (R \supset S)) \supset ((P \vee (R \supset S)) \wedge (Q \vee (R \supset S))))]$	
$2a$	$[P \wedge Q, R \supset S]$	
$2b$	$[\neg((P \vee (R \supset S)) \wedge (Q \vee (R \supset S)))]$	$1, NotImplies + Or$
3	$[P, R \supset S]; [Q, R \supset S]$	$2a, And$
4	$[\neg(P \vee (R \supset S)), \neg(Q \vee (R \supset S))]$	$2b, NotAnd$
$5a$	$[\neg P, \neg(Q \vee (R \supset S))]$	
$5b$	$[\neg(R \supset S), \neg(Q \vee (R \supset S))]$	$4, NotOr$

Cut (Gerhard Gentzen 1934, Alan Robinson 1968)

- Cut/Resolution on A: $\langle [B, A]; [\neg A, C] \rangle \to [B, C]$. From (MP).
- One-literal/Unit cut propagation: If $[L]$ in clause set S, delete clauses of S containing L (including unit clause) and occurrences in S of complement $\overline{L}$.

1	$[\neg(((P \wedge Q) \vee (R \supset S)) \supset ((P \vee (R \supset S)) \wedge (Q \vee (R \supset S))))]$	
$2a$	$[P \wedge Q, R \supset S]$	
$2b$	$[\neg((P \vee (R \supset S)) \wedge (Q \vee (R \supset S)))]$	$1, \mathit{NotImplies} + \mathit{Or}$
3	$[P, R \supset S]; [Q, R \supset S]$	$2a, \mathit{And}$
4	$[\neg(P \vee (R \supset S)), \neg(Q \vee (R \supset S))]$	$2b, \mathit{NotAnd}$
$5a$	$[\neg P, \neg(Q \vee (R \supset S))]$	
$5b$	$[\neg(R \supset S), \neg(Q \vee (R \supset S))]$	$4, \mathit{NotOr}$
6	$[\neg P, \neg Q]; [\neg P, \neg(R \supset S)]$	$5a, \mathit{NotOr}$

Cut (Gerhard Gentzen 1934, Alan Robinson 1968)

- Cut/Resolution on A: $\langle [B, A]; [\neg A, C] \rangle \to [B, C]$. From (MP).
- One-literal/Unit cut propagation: If $[L]$ in clause set S, delete clauses of S containing L (including unit clause) and occurrences in S of complement $\overline{L}$.

1	$[\neg(((P \wedge Q) \vee (R \supset S)) \supset ((P \vee (R \supset S)) \wedge (Q \vee (R \supset S))))]$	
$2a$	$[P \wedge Q, R \supset S]$	
$2b$	$[\neg((P \vee (R \supset S)) \wedge (Q \vee (R \supset S)))]$	$1, \mathit{NotImplies} + \mathit{Or}$
3	$[P, R \supset S]; [Q, R \supset S]$	$2a, \mathit{And}$
4	$[\neg(P \vee (R \supset S)), \neg(Q \vee (R \supset S))]$	$2b, \mathit{NotAnd}$
$5a$	$[\neg P, \neg(Q \vee (R \supset S))]$	
$5b$	$[\neg(R \supset S), \neg(Q \vee (R \supset S))]$	$4, \mathit{NotOr}$
6	$[\neg P, \neg Q]; [\neg P, \neg(R \supset S))]$	$5a, \mathit{NotOr}$
7	$[\neg(R \supset S), \neg Q]; [\neg(R \supset S)]$	$5b, \mathit{NotOr}$

Cut (Gerhard Gentzen 1934, Alan Robinson 1968)

- Cut/Resolution on A: $\langle [B, A]; [\neg A, C] \rangle \to [B, C]$. From (MP).
- One-literal/Unit cut propagation: If $[L]$ in clause set S, delete clauses of S containing L (including unit clause) and occurrences in S of complement $\overline{L}$.

1	$[\neg(((P \wedge Q) \vee (R \supset S)) \supset ((P \vee (R \supset S)) \wedge (Q \vee (R \supset S))))]$	
$2a$	$[P \wedge Q, R \supset S]$	
$2b$	$[\neg((P \vee (R \supset S)) \wedge (Q \vee (R \supset S)))]$	$1, NotImplies + Or$
3	$[P, R \supset S]; [Q, R \supset S]$	$2a, And$
4	$[\neg(P \vee (R \supset S)), \neg(Q \vee (R \supset S))]$	$2b, NotAnd$
$5a$	$[\neg P, \neg(Q \vee (R \supset S))]$	
$5b$	$[\neg(R \supset S), \neg(Q \vee (R \supset S))]$	$4, NotOr$
6	$[\neg P, \neg Q]; [\neg P, \neg(R \supset S)]$	$5a, NotOr$
7	$[\neg(R \supset S), \neg Q]; [\neg(R \supset S)]$	$5b, NotOr$
8	$[P, \neg Q]$	$3a, 7a, Cut\ R \supset S$

Cut (Gerhard Gentzen 1934, Alan Robinson 1968)

- Cut/Resolution on A: $\langle[B, A]; [\neg A, C]\rangle \rightarrow [B, C]$. From (MP).
- One-literal/Unit cut propagation: If $[L]$ in clause set S, delete clauses of S containing L (including unit clause) and occurrences in S of complement $\overline{L}$.

1	$[\neg(((P \wedge Q) \vee (R \supset S)) \supset ((P \vee (R \supset S)) \wedge (Q \vee (R \supset S))))]$	
$2a$	$[P \wedge Q, R \supset S]$	
$2b$	$[\neg((P \vee (R \supset S)) \wedge (Q \vee (R \supset S)))]$	$1, NotImplies + Or$
3	$[P, R \supset S]; [Q, R \supset S]$	$2a, And$
4	$[\neg(P \vee (R \supset S)), \neg(Q \vee (R \supset S))]$	$2b, NotAnd$
$5a$	$[\neg P, \neg(Q \vee (R \supset S))]$	
$5b$	$[\neg(R \supset S), \neg(Q \vee (R \supset S))]$	$4, NotOr$
6	$[\neg P, \neg Q]; [\neg P, \neg(R \supset S))]$	$5a, NotOr$
7	$[\neg(R \supset S), \neg Q]; [\neg(R \supset S)]$	$5b, NotOr$
8	$[P, \neg Q]$	$3a, 7a, Cut\ R \supset S$
9	$[\neg Q]$	$6a, 8, Cut\ P$

Cut (Gerhard Gentzen 1934, Alan Robinson 1968)

- Cut/Resolution on A: $\langle[B, A]; [\neg A, C]\rangle \to [B, C]$. From (MP).
- One-literal/Unit cut propagation: If $[L]$ in clause set S, delete clauses of S containing L (including unit clause) and occurrences in S of complement $\overline{L}$.

1	$[\neg(((P \wedge Q) \vee (R \supset S)) \supset ((P \vee (R \supset S)) \wedge (Q \vee (R \supset S))))]$	
$2a$	$[P \wedge Q, R \supset S]$	
$2b$	$[\neg((P \vee (R \supset S)) \wedge (Q \vee (R \supset S)))]$	$1, \mathit{NotImplies} + \mathit{Or}$
3	$[P, R \supset S]; [Q, R \supset S]$	$2a, \mathit{And}$
4	$[\neg(P \vee (R \supset S)), \neg(Q \vee (R \supset S))]$	$2b, \mathit{NotAnd}$
$5a$	$[\neg P, \neg(Q \vee (R \supset S))]$	
$5b$	$[\neg(R \supset S), \neg(Q \vee (R \supset S))]$	$4, \mathit{NotOr}$
6	$[\neg P, \neg Q]; [\neg P, \neg(R \supset S))]$	$5a, \mathit{NotOr}$
7	$[\neg(R \supset S), \neg Q]; [\neg(R \supset S)]$	$5b, \mathit{NotOr}$
8	$[P, \neg Q]$	$3a, 7a, \mathit{Cut}\ R \supset S$
9	$[\neg Q]$	$6a, 8, \mathit{Cut}\ P$
10	$[R \supset S]$	$3b, 9, \mathit{Unit\ Cut}\ Q$

Cut (Gerhard Gentzen 1934, Alan Robinson 1968)

- Cut/Resolution on A: $\langle [B, A]; [\neg A, C] \rangle \to [B, C]$. From (MP).
- One-literal/Unit cut propagation: If $[L]$ in clause set S, delete clauses of S containing L (including unit clause) and occurrences in S of complement $\overline{L}$.

1	$[\neg(((P \wedge Q) \vee (R \supset S)) \supset ((P \vee (R \supset S)) \wedge (Q \vee (R \supset S))))]$	
$2a$	$[P \wedge Q, R \supset S]$	
$2b$	$[\neg((P \vee (R \supset S)) \wedge (Q \vee (R \supset S)))]$	$1, NotImplies + Or$
3	$[P, R \supset S]; [Q, R \supset S]$	$2a, And$
4	$[\neg(P \vee (R \supset S)), \neg(Q \vee (R \supset S))]$	$2b, NotAnd$
$5a$	$[\neg P, \neg(Q \vee (R \supset S))]$	
$5b$	$[\neg(R \supset S), \neg(Q \vee (R \supset S))]$	$4, NotOr$
6	$[\neg P, \neg Q]; [\neg P, \neg(R \supset S)]$	$5a, NotOr$
7	$[\neg(R \supset S), \neg Q]; [\neg(R \supset S)]$	$5b, NotOr$
8	$[P, \neg Q]$	$3a, 7a, Cut\ R \supset S$
9	$[\neg Q]$	$6a, 8, Cut\ P$
10	$[R \supset S]$	$3b, 9, Unit\ Cut\ Q$
11	$[]$	$7b, 10, Cut\ R \supset S$

Split (George Boole 1854, Claude Shannon 1948)

- Check satisfiability of A by testing that of $A \wedge L$ and of $A \wedge \overline{L}$.
- Splitting on L: If clause set S has clauses with L and clauses with $\overline{L}$, replace S with: clause set S_1 where clauses containing L are removed, and occurrences of $\overline{L}$ are deleted, and clause set S_2 where clauses containing $\overline{L}$ are removed and occurrences of L are deleted.

Split (George Boole 1854, Claude Shannon 1948)

- Check satisfiability of A by testing that of $A \wedge L$ and of $A \wedge \overline{L}$.
- Splitting on L: If clause set S has clauses with L and clauses with $\overline{L}$, replace S with: clause set S_1 where clauses containing L are removed, and occurrences of $\overline{L}$ are deleted, and clause set S_2 where clauses containing $\overline{L}$ are removed and occurrences of L are deleted.

1 $[\langle [P, Q, R]; [P, \neg R]; [\neg Q, R]; [\neg P, R]; [\neg P, \neg Q, \neg R] \rangle]$

2 $[\langle [Q, R]; [\neg R]; [\neg Q, R] \rangle, \langle [\neg Q, R]; [R]; [\neg Q, \neg R] \rangle]$ *Split on P*

Split (George Boole 1854, Claude Shannon 1948)

- Check satisfiability of A by testing that of $A \wedge L$ and of $A \wedge \overline{L}$.
- Splitting on L: If clause set S has clauses with L and clauses with $\overline{L}$, replace S with: clause set S_1 where clauses containing L are removed, and occurrences of $\overline{L}$ are deleted, and clause set S_2 where clauses containing $\overline{L}$ are removed and occurrences of L are deleted.

1 $[\langle [P, Q, R]; [P, \neg R]; [\neg Q, R]; [\neg P, R]; [\neg P, \neg Q, \neg R] \rangle]$

2 $[\langle [Q, R]; [\neg R]; [\neg Q, R] \rangle, \langle [\neg Q, R]; [R]; [\neg Q, \neg R] \rangle]$ *Split on P*

Other rules:

- Affirmative-negative: If some literal L occurs only positively/ only negatively in clause set S, delete clauses in S with L.
- Subsumption: If clause set S has clause C_1 subsuming C_2 (every literal in C_1 occurs in C_2), delete clause C_2 from S.

(Martin Davis-Hilary Putnam 1960) refutation system

Preliminary steps: Put in block form, remove repetitions from clauses, order literals, delete *False*, delete clauses containing *True*, delete clauses containing literal and its complement.

Definition

- *A refutation from* $\{A_1, \ldots, A_n\}$ *consists of beginning with the block* $[\langle [A_1]; \ldots; [A_n] \rangle]$*, and applying rules.*
- *If the empty clause* $[]$ *appears in a clause set, it is closed.*
- *A refutation is closed if all clause sets are closed.*
- *A proof of* A *(written* $\vdash_{pDP} A$*) is a closed refutation from* $\{\neg A\}$*.*
- Th *is consistent if no closed refutation from* Th*.*

Theorem (Fitting, Proposition 4.4.3, page 89)

(Unit Propagation) preserves satisfiability.

- Ignoring the unaffected clause sets, suppose $S = \langle [L]; [L, K_1]; \ldots; [L, K_n]; [\overline{L}, N_1]; \ldots; [\overline{L}, N_m]; [O_1]; \ldots; [O_l] \rangle$ with neither L nor $\overline{L}$ appearing in the K_i, N_i, O_i.
- The resolvent is $\langle [N_1]; \ldots; [N_m]; [O_1]; \ldots; [O_l] \rangle$.
- First suppose v makes every clause of S true. Then $v(L) = v(N_i) = v(O_i) = true$. So resolvent is satisfiable.
- On the other hand, if the resolvent is satisfiable, every clause in it is made true by some v. Define $v'(L) = true$, $v'(P) = v(P)$ otherwise. v' makes every clause of S true.

Exercise

Show that other rules preserve satisfiability.

PL decidability (Ludwig Wittgenstein, Emil Post 1921)

Theorem (Decidability, Fitting, Theorem 4.4.6)

The Davis-Putnam refutation procedure for A terminates establishing whether A is a tautology or not.

- If a nonempty clause set does not contain the empty clause, then some rule is applicable. Choose order.
- So a terminating refutation attempt ends with satisfiability (some clause set is empty) or unsatisfiability (every clause set has empty clause). Implicit backtracking, $O(n2^n)$.
- Every rule reduces the number of literals except for (Subsumption), which removes clauses, so terminating.
- A is a theorem ($\vdash_{pDP} A$) iff starting block is not satisfiable.

Question

Can these ideas be extended to ZOL?

PHP_2 (Cook and Reckhow 1979, Haken 1985)

Given 3 pigeons and 2 pigeonholes.
Let P_{ij} stand for pigeon i in pigeonhole j (Fitting, Section 4.5).

Propositional pigeonhole principle:

$$((P_{11} \vee P_{12}) \wedge (P_{21} \vee P_{22}) \wedge (P_{31} \vee P_{32}))$$
$$\supset ((P_{11} \wedge P_{21}) \vee (P_{21} \wedge P_{31}) \vee (P_{11} \wedge P_{31})$$
$$\vee (P_{12} \wedge P_{22}) \vee (P_{22} \wedge P_{32}) \vee (P_{12} \wedge P_{32}))$$

PHP_2 (Cook and Reckhow 1979, Haken 1985)

Given 3 pigeons and 2 pigeonholes.
Let P_{ij} stand for pigeon i in pigeonhole j (Fitting, Section 4.5).

Propositional pigeonhole principle:

$$((P_{11} \vee P_{12}) \wedge (P_{21} \vee P_{22}) \wedge (P_{31} \vee P_{32}))$$
$$\supset ((P_{11} \wedge P_{21}) \vee (P_{21} \wedge P_{31}) \vee (P_{11} \wedge P_{31})$$
$$\vee (P_{12} \wedge P_{22}) \vee (P_{22} \wedge P_{32}) \vee (P_{12} \wedge P_{32}))$$

Negated, negations pushed to literals:

$$(P_{11} \vee P_{12}) \wedge (P_{21} \vee P_{22}) \wedge (P_{31} \vee P_{32})$$
$$\wedge (\neg P_{11} \vee \neg P_{21}) \wedge (\neg P_{21} \vee \neg P_{31}) \wedge (\neg P_{11} \vee \neg P_{31})$$
$$\wedge (\neg P_{12} \vee \neg P_{22}) \wedge (\neg P_{22} \vee P_{32}) \wedge (\neg P_{12} \vee \neg P_{32})$$

PHP_2 (Cook and Reckhow 1979, Haken 1985)

Given 3 pigeons and 2 pigeonholes.
Let P_{ij} stand for pigeon i in pigeonhole j (Fitting, Section 4.5).

Propositional pigeonhole principle:

$$((P_{11} \vee P_{12}) \wedge (P_{21} \vee P_{22}) \wedge (P_{31} \vee P_{32}))$$
$$\supset ((P_{11} \wedge P_{21}) \vee (P_{21} \wedge P_{31}) \vee (P_{11} \wedge P_{31})$$
$$\vee (P_{12} \wedge P_{22}) \vee (P_{22} \wedge P_{32}) \vee (P_{12} \wedge P_{32}))$$

Negated, negations pushed to literals:

$$(P_{11} \vee P_{12}) \wedge (P_{21} \vee P_{22}) \wedge (P_{31} \vee P_{32})$$
$$\wedge(\neg P_{11} \vee \neg P_{21}) \wedge (\neg P_{21} \vee \neg P_{31}) \wedge (\neg P_{11} \vee \neg P_{31})$$
$$\wedge(\neg P_{12} \vee \neg P_{22}) \wedge (\neg P_{22} \vee P_{32}) \wedge (\neg P_{12} \vee \neg P_{32})$$

Negation, block form, $O(n^3)$ clauses for PHP_n:

$$\langle [P_{11}, P_{12}]; [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{11}, \neg P_{21}]; [\neg P_{21}, \neg P_{31}]$$
$$; [\neg P_{11}, \neg P_{31}]; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle$$

Negation, block form:

$\langle [P_{11}, P_{12}]; [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{11}, \neg P_{21}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{11}, \neg P_{31}]; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle$

Negation, block form:

$\langle [P_{11}, P_{12}]; [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{11}, \neg P_{21}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{11}, \neg P_{31}]; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle$

(Splitting) on P_{11} (first pigeon in first pigeonhole or not):

$[\langle [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{21}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{31}]; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle,$
$\langle [P_{12}]; [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle]$

Negation, block form:

$\langle [P_{11}, P_{12}]; [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{11}, \neg P_{21}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{11}, \neg P_{31}]; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle$

(Splitting) on P_{11} (first pigeon in first pigeonhole or not):

$[\langle [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{21}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{31}]; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle,$
$\langle [P_{12}]; [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle]$

(Negative) on P_{12} in first clause set:

$[\langle [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{21}]; [\neg P_{21}, \neg P_{31}]; [\neg P_{31}]; [\neg P_{22}, \neg P_{32}] \rangle, \langle \ldots \rangle$

Negation, block form:

$\langle [P_{11}, P_{12}]; [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{11}, \neg P_{21}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{11}, \neg P_{31}]; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle$

(Splitting) on P_{11} (first pigeon in first pigeonhole or not):

$[\langle [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{21}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{31}]; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle,$
$\langle [P_{12}]; [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle]$

(Negative) on P_{12} in first clause set:

$[\langle [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{21}]; [\neg P_{21}, \neg P_{31}]; [\neg P_{31}]; [\neg P_{22}, \neg P_{32}] \rangle, \langle \ldots \rangle$

(Unit) on $\neg P_{21}$ (excluding second pigeon) in first set:

$[\langle [P_{22}]; [P_{31}, P_{32}]; [\neg P_{31}]; [\neg P_{22}, \neg P_{32}] \rangle, \quad \langle \ldots \rangle]$

Propositional pigeonhole principle

Negation, block form:

$\langle [P_{11}, P_{12}]; [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{11}, \neg P_{21}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{11}, \neg P_{31}]; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle$

(Splitting) on P_{11} (first pigeon in first pigeonhole or not):

$[\langle [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{21}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{31}]; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle,$
$\langle [P_{12}]; [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{21}, \neg P_{31}]$
$; [\neg P_{12}, \neg P_{22}]; [\neg P_{22}, \neg P_{32}]; [\neg P_{12}, \neg P_{32}] \rangle]$

(Negative) on P_{12} in first clause set:

$[\langle [P_{21}, P_{22}]; [P_{31}, P_{32}]; [\neg P_{21}]; [\neg P_{21}, \neg P_{31}]; [\neg P_{31}]; [\neg P_{22}, \neg P_{32}] \rangle, \langle \ldots \rangle$

(Unit) on $\neg P_{21}$ (excluding second pigeon) in first set:

$[\langle [P_{22}]; [P_{31}, P_{32}]; [\neg P_{31}]; [\neg P_{22}, \neg P_{32}] \rangle, \quad \langle \ldots \rangle]$

(Unit) on P_{22} (second in second) in first set, closure ahead:

$[\langle [P_{31}, P_{32}]; [\neg P_{31}]; [P_{32}]; [\neg P_{32}] \rangle, \quad \langle \ldots \rangle]$

Similarly second clause set.

$O(c^n),\ c > 1$ proof steps lower bound for PHP_n (Haken 1985).